CMI undergraduate entrance exam on May 2, 2026

Draft solutions from next page

Send any comments to ugadmit2026@cmi.ac.in by May 24, 2026.

Comments may not get a reply but all comments will be carefully read and considered.

To learn higher mathematics, it is vital to absorb new concepts, follow given rigorous reasoning, and build upon it. Questions like A11 to A16 and B5, which supply definitions/partial proofs/hints, are designed to mimic this process and test your ability to engage with it.

Unless specified otherwise, in this exam all numbers are real and "function" means a function whose codomain as well as domain is the set of all real numbers or an implied subset. You may use the following information if you find it relevant.

$$\text{prime factorization of } 2026 = 2 \times 1013$$

$$\ln(\sqrt{2}) \approx 0.347 \qquad \ln(\sqrt{3}) \approx 0.549 \qquad \ln(\sqrt{5}) \approx 0.805$$

$$\tan^{-1}(2) \approx 1.107 \qquad \tan^{-1}(3) \approx 1.249 \qquad \tan^{-1}(5) \approx 1.373$$

Instructions for Part A

- **Part A is worth 40 points.** It has 15 questions, which are numbered 2 to 16 for technical reasons. *There is no negative marking.*
- For each of the Part A questions, **type in your answer** as directed. For the questions where the answer need not be a single nonnegative integer, the format of the answer is explained with the question and an example is given in **blue**. *Read the instructions carefully.*
- **Part A will be used for screening.** Part B is assured to be graded if you score at least 21 in part A. The cutoff may be lower at the discretion of CMI.

Instructions for Part B

- **Part B has 5 problems worth a total of 80 points.** Solve each part B problem on the designated pages in the answer booklet.
- **Clearly explain your entire reasoning**. No credit will be given without correct reasoning. You may solve a later part of a problem by assuming some previous part(s), even if you could not do the earlier part(s).
- You are advised to spend at least 2 hours on part B.

Part A questions

There are 15 questions numbered 2 to 16. Question numbering deliberately starts at 2.

Information for questions (2) to (4)

Two friends A and B purchased three identical boxes, each containing 12 assorted chocolates: 5 dark chocolates, 4 milk chocolates and 3 white chocolates. Answer questions (2) to (4) as per the given instruction. Here "random" means that each possible subset of chocolates is equally likely to be picked.

Instruction for (2) to (4)

Express each probability as $\frac{a}{b}$ with a and b coprime positive integers. Type the values of a, b separated by commas with no gaps. E.g., for probability $\frac{6}{10}$, you should type 3,5 as your answer.

Questions

(2) A picks 2 chocolates at random from the first box. What is the probability that both are dark? [1 point]

(3) B picks 3 chocolates at random from the second box. What is the probability that he picks exactly one of each type (one dark, one milk, and one white)? [1 point]

(4) The third box is for a friend C, who is not with A and B but is on the phone with them. C asks B to pick 3 chocolates at random from the third box. Since C likes white chocolates, he asks B whether there is at least one white among the three chocolates B has picked from the third box. B, who can see the chocolates he picked for C, says yes. What is the probability that the other two are dark? [3 points]

Solutions

(2) $\binom{5}{2}/\binom{12}{2} = \frac{5}{33}$. Answer: 5,33

(3) $(5 \times 4 \times 3)/\binom{12}{3} = \frac{3}{11}$. Answer: 3,11

(4) Number of ways to get 1 white and 2 dark $= 3 \times 10 = 30$. Number of ways to get at least one white $= \binom{12}{3} -$ (number of ways to get no white) $= 220 - \binom{9}{3} = 220 - 84 = 136$. So the required probability is $30/136 = 15/68$. Answer: 15,68

(5) Find the number of distinct complex numbers z such that $z^{20} = \overline{z}^{26}$, where $\overline{z}$ denotes the complex conjugate of z. Type the word infinite if that is the answer. [3 points]

Solution

Let $z = re^{\theta i}$ with $r \geq 0$. We have $\overline{z} = re^{-\theta i}$. The equation becomes $r^{20}e^{20\theta i} = r^{26}e^{-26\theta i}$. We get $r = 0$, i.e., $z = 0$ as one solution. If $r \neq 0$, then we have $r^{-6}e^{46\theta i} = 1$, forcing $r = 1 = e^{46\theta i}$. So there are 46 nonzero solutions, namely $e^{\frac{2\pi k i}{46}}$ with $k = 0, 1, \ldots, 45$. Altogether there are 47 solutions.

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(6) In triangle ABC we have $\angle B = 20°$ and $\angle C = 26°$. Let M be the midpoint of AB and N the midpoint of AC. Let BN and CM intersect in a point G. Let P be the midpoint of BG and Q the midpoint of the CG.

Suppose quadrilateral $MNQP$ has area 100. Find the *integer* closest to the area of triangle ABC. E.g., if the answer is any of $e, 3, \pi$, then enter 3. For answer 3.5 enter 4. [3 points]

Solution

G is the centroid, i.e. the intersection point of all three medians, and it divides each median in ratio $2:1$. Therefore (i) the area of triangle GBC is a third of that of triangle ABC, and (ii) $BP = PG = GN$ and $CQ = QG = GM$, making $MNQP$ a parallelogram as its diagonals bisect each other. By (ii), area of triangle $GPQ = \frac{1}{4}$ (area of $MNQP$). Also area of triangle $GPQ = \frac{1}{4}$ (area of triangle GBC) by the midpoint theorem. Thus parallelogram $MNQP$ and triangle GBC have equal areas. Putting together, the area of triangle $ABC = 3$(area of $MNQP$) $= 300$. The given angles are irrelevant.

Information for questions (7) to (9)

Consider the following three non-collinear points given in terms of their x, y, z coordinates: $P = (2, 3, 4), Q = (10, 10, 10), R = (25, 20, 26)$.

Questions

(7) Suppose a parallelogram has P, Q, R as three of its four vertices. How many such parallelograms are there? Type the number or the word infinite if that is the answer. [1 point]

(8) Suppose $PQRS$ is a parallelogram for a point $S = (p, q, r)$. Find one such point S. You should enter your answer as integers closest to p, q, r in that order, separated by commas with no gaps. E.g., for $(p, q, r) = (-10.2, 8.5, 22)$, you would type $-10, 9, 22$ as the answer. [2 points]

(9) Consider the lines PQ, PR and QR. These lines divide the remaining points in the plane formed by P, Q, R into seven regions. Pick a point S in one of these regions. Let

$A =$ the number of these regions for which $PQRS$ is a convex quadrilateral,

$B =$ the number of these regions for which $PQRS$ is a quadrilateral intersecting itself, and

$C =$ the number of regions for which $PQRS$ is a nonconvex and non-self-intersecting.

Type values of the numbers A, B, C separated by commas, with no gaps. E.g., 2,3,2 is written in the correct format. [2 points]

Solutions

(7) 3, depending on which of the three sides of triangle PQR becomes a diagonal.

(8) The diagonals of a $PQRS$ bisect each other, say at point $M = (a, b, c)$. Work in terms of coordinates and use the midpoint formula to get $(2a, 2b, 2c) = P + R = (27, 23, 30) = Q + S$, so $S = (17, 13, 20)$. Answer: 17,13,20

(9) Note that the order of vertices is specified. Draw a picture to see that $C = 4$ (when S is inside triangle PQR or in an open cone opposite one of the vertices), $B = 2$ (when S is in one of the remaining regions having either segment PQ or segment QR as a boundary) and $A = 1$ (when S is in the remaining region having segment PR a boundary). Answer: 1,2,4

(10) Evaluate

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{k^2+n^2}\right)$$

and write your answer correct to three decimals. E.g., for $(e-2)$ write 0.718. Write DNE if no real number is the limit. Hint: relate the limit to an integral. [3 points]

Solution

Following the hint, observe that $\sum_{k=1}^{n}\frac{k}{k^2+n^2}=\frac{1}{n}\sum_{k=1}^{n}\frac{k/n}{(k/n)^2+1}$ is the Riemann sum for $\frac{x}{1+x^2}$ on the interval $[0,1]$ using subdivision into n equal parts and using right hand endpoints. By the fundamental theorem of calculus, the required limit is $\frac{1}{2}[\ln(1+x^2)]_{x=0}^{x=1}=\frac{1}{2}\ln 2$, which is 0.347 upto 3 decimals. Answer: 0.347 or .347 (partial credit for 0.346)

Information for question (11)

Consider the equation E_1 where a_0, a_1, a_2, a_3, a_4 are *integers* with $a_4 \neq 0$.

Equation E_1: $\quad a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0.$

The proof of a standard result is sketched below. Complete it as instructed. Note that LHS means the left hand side of an equation and RHS the right hand side.

Suppose $x=\frac{b}{c}$ is a solution of E_1, where b and c are coprime integers, i.e., $gcd(b,c)=1$. Substitute $\underline{\quad 1 \quad}$ into equation E_1 and multiply both sides by $\underline{\quad 2 \quad}$ to get an equality E_2 of integers. In E_2 subtract the first term on the LHS from both sides to get a third equation E_3. We see that the LHS of equation E_3 is divisible by $\underline{\quad 3 \quad}$ and therefore so is the RHS. This forces the coefficient $\underline{\quad 4 \quad}$ to be divisible by $\underline{\quad 5 \quad}$.

A parallel argument shows, by subtracting the $\underline{\quad 6 \quad}$ term on the LHS of equation E_2, that $\underline{\quad 7 \quad}$ is divisible by $\underline{\quad 8 \quad}$.

Options for the blanks

A. 0 B. $\frac{c}{b}$ C. $\frac{b}{c}$ D. b E. b^2 F. b^4 G. c H. c^2 I. c^4

J. a_0 K. a_1 L. a_2 M. a_3 N. a_4 O. second P. middle Q. last

Question

(11) Complete the proof by typing in a sequence of 8 letters indicating the correct options to fill in the numbered blanks 1 to 8. Do not use any spaces, full stop or any other punctuation. E.g., **ABACDIJK** is written in the correct format. [4 points]

Solution

Substitute b/c into equation E_1 and multiply both sides by c^4 to get an equality E_2 of integers. In E_2 subtract the first term on the LHS from both sides to get a third equation E_3. We see that the LHS of equation E_3 is divisible by c and therefore so is the RHS. This forces the coefficient a_4 to be divisible by c. A parallel argument shows, by subtracting the last term on the LHS of equation E_2, that a_0 is divisible by b. Answer: CIGNGQJD

Information for question 12

Let $x_1, x_2, \ldots, x_n$ be positive numbers. Define $A = (x_1 + \cdots + x_n)/n$ and $G = \sqrt[n]{x_1 \cdots x_n}$.

A proof that $A \geq G$: For any real r we have the inequality $(*)$ $e^{r-1} \geq r$. Reason: We have equality in $(*)$ when $r = \underline{\quad 1 \quad}$. Let $f(r) = e^{r-1} - r$. For values of r lower than the previous blank, $\underline{\quad 2 \quad}$ is $\underline{\quad 3 \quad}$. And for higher values of r, $\underline{\quad 4 \quad}$ is $\underline{\quad 5 \quad}$.

Substitute $\frac{x_i}{A}$ for r in $(*)$ for $i = 1, \ldots, n$. $\underline{\quad 6 \quad}$ the resulting n inequalities. The left hand side of the combined inequality evaluates to $\underline{\quad 7 \quad}$. This gives $A \geq G$.

The reasoning also shows that $A = G$ precisely when the value of each $\frac{x_i}{A}$ is $\underline{\quad 8 \quad}$.

Options for the blanks

A. $f(r)$

B. $f'(r)$

C. $f''(r)$

D. positive

E. negative

F. e^{A-1}

G. 1

H. A

I. G

J. Add

K. Multiply

L. Divide

Question

(12) Complete the proof by typing in a sequence of 8 letters indicating the correct options to fill in the numbered blanks 1 to 8. Do not use any spaces, full stop or any other punctuation. E.g., **ABACDIJK** is written in the correct format. [4 points]

Solution

For any real r we have the inequality $(*)$ $e^{r-1} \geq r$. Reason: We have equality in $(*)$ when $r = 1$. Let $f(r) = e^{r-1} - r$. For values of r lower than the previous blank, $f'(r)$ is negative. And for higher values of r, $f'(r)$ is positive.

Substitute $\frac{x_i}{A}$ for r in $(*)$ for $i = 1, \ldots, n$. Multiply the resulting n inequalities. The left hand side of the combined inequality evaluates to 1. This gives $A \geq G$.

The reasoning also shows that $A = G$ precisely when the value of each $\frac{x_i}{A}$ is 1.

Answer: GBEBDKGG

Information for question 13

We revisit the reasoning in question 12 in a more general situation. Let $x_1, x_2, \ldots, x_n$ be positive numbers. Also let $w_1, \ldots, w_n$ be positive numbers that add to 1. **Claim**: $A' \geq G'$, where

$A' = w_1 x_1 + \cdots + w_n x_n$ and $G' = {x_1}^{w_1} \cdots x_n^{w_n}$.

Proof: Substitute $\frac{x_i}{A'}$ for r in $e^{r-1} \geq r$ for $i = 1, \ldots, n$. Then, in the i^{th} resulting inequality, we do the following operation to both sides. $\underline{\quad 1 \quad}$. Now similar algebra as in the previous question proves the claim.

Does the criterion for $A' = G'$ that you found in the last sentence of question 12 remain valid? $\underline{\quad 2 \quad}$ Use letter Y (for Yes) or N (for No) in your answer.

Further discussion: Let us now see how we might have arrived at the clever choice of substituting $\frac{x_i}{A'}$ for r in $e^{r-1} \geq r$. The proof of $A' \geq G'$ depended crucially on this choice. If we instead substitute x_i for r for $i = 1, \ldots, n$, then similar algebra gives us the inequality $\underline{\quad 3 \quad} \geq G'$. Notice that we would like to replace the LHS by A' in the preceding inequality, but we cannot do that in general. However, we may replace LHS by A' if the value of A' is $\underline{\quad 4 \quad}$. Now notice that we can always reduce to the situation in the previous sentence because if we scale all x_i by a positive constant c, then each side of the claimed inequality $\underline{\quad 5 \quad}$.

Options for the blanks

A. 1
B. Add w_i
C. Multiply by w_i
D. 0
E. Divide by w_i
F. Raise to power w_i
G. G'
H. is unchanged
I. is also scaled by c
J. A'
K. $e^{A'-1}$
L. $e^{A'}$

Question

(13) Complete the proof by typing in a sequence of 5 letters indicating the correct options to fill in the numbered blanks 1 to 5. Do not use any spaces, full stop or any other punctuation. E.g., **AYCKA** is written in the correct format. [3 points]

Solution

Proof: Substitute $\frac{x_i}{A'}$ for r in $e^{r-1} \geq r$ for $i = 1, \ldots, n$. Then, in the i^{th} resulting inequality, we do the following operation to both sides: raise to power w_i. Now similar algebra as in the previous question proves the claim. The criterion for $A' = G'$ in the last sentence of question 12 does remain valid (each $\frac{x_i}{A'} = 1$, so all x_i equal, just like before).

Further discussion: Let us now see how we might have arrived at the clever choice of substituting $\frac{x_i}{A'}$ for r in $e^{r-1} \geq r$. The proof of $A' \geq G'$ depended crucially on this choice. If we instead substitute x_i for r for $i = 1, \ldots, n$, then similar algebra gives us the inequality $e^{A'-1} \geq G'$. Notice that we would like to replace the LHS by A' in the preceding inequality, but we cannot do that in general. However, we may replace LHS by A' if the value of A' is 1. Now notice that we can always reduce to the situation in the previous sentence because if we scale all x_i by a positive constant c, then each side of the claimed inequality is also scaled by c. Answer: FYKAI

Information for question 14

Let $A = \{1,2,3,4,5,6,7,8,9\}$. For a subset S of A, let $\Sigma(S)$ = the sum of elements of S. E.g. $\Sigma(\{2,3\}) = 5$, $\Sigma(A) = 45$ and $\Sigma(\text{empty set}) = 0$.

Let us find N = the number of subsets S with $\Sigma(S)$ divisible by 4. For this, remove 1 and 2 from A to get $B = \{3,4,5,6,7,8,9\}$. Now for any given subset T of B, consider the four numbers

$$\Sigma(T), \Sigma(T \cup \{1\}), \Sigma(T \cup \{2\}), \Sigma(T \cup \{1,2\}).$$

This collection of numbers, considered modulo 4, does not depend on T. Using this we get $N =$ ____.

What is the next smallest number $k > 4$ for which same idea can be used to find M = number of subsets S with $\Sigma(S)$ divisible by k? We have $k =$ ____, and it works by removing the following elements from A to get a set C: ____ (List in increasing order numbers removed from A, e.g., for removing 2, 4 and 6 from A, type 246 without any gaps or commas. If multiple lists are possible, give one with the least sum.) To find M, for any subset T of C, we construct a list of subsets of A by suitably adding the removed elements to T and see that modulo k, the subset sums of this list show uniform behavior independent of T. We get $M =$ ____.

Question

(14) Complete the solution by typing in four numbers separated by commas. E.g., if $N = 100, k = 5$, removed list is 2,4,6, and $M = 50$, your answer would be 100,5,246,50 without any gaps or other punctuation. [4 points]

Solution

N = a quarter of the number of all subsets of A, i.e., $N = \frac{2^9}{4} = 128$.

What is the next smallest number $k > 4$ for which same idea can be used to find M = number of subsets S with $\Sigma(S)$ divisible by k? We have $k = 8$, and it works by removing the elements $1, 2, 4$ from A to get a set $C = \{3,5,6,7,8,9\}$. To find M, for any subset T of C, we construct a list of 8 subsets of A as follows: add to T all eight subsets of $\{1,2,4\}$ one by one and see that modulo 8, the subset sums of this list show uniform behavior independent of T: exactly one of the eight subset sums is 1 mod 8, exactly one is 2 mod 8 and so on. So M = an eighth of the number of all subsets of A, which is $\frac{2^9}{8} = 64$. Answer: 128,8,124,64

Information for questions 15 and 16

Let g be a function from a nonempty set Y to a set Z. The three properties A, B, C stated below are equivalent. Complete the given proof of this fact. Here $\circ$ denotes function composition.

A) There is a function f from Z to Y such that $f \circ g = id_Y =$ function defined by $id_Y(y) = y$ for each element y of Y .

B) For any set X and functions h_1, h_2 from X to the set Y, if $g \circ h_1 = g \circ h_2$, then $h_1 = h_2$.

C) The function g is one-to-one.

Proof of equivalence: Suppose property A is true. If $g \circ h_1 = g \circ h_2$, then applying the function $\underline{\quad 1 \quad}$ on the $\underline{\quad 2 \quad}$ of both sides of this equation gives $h_1 = h_2$ by $\underline{\quad 3 \quad}$ of function composition, proving B.

Suppose B is true. To prove C, suppose (i) $\underline{\quad 4 \quad}$ where y_1 is in Y and y_2 is in Y. Take $X = \{a\}$, a set with a single element a. Define functions h_1, h_2 from this set X to the set Y as follows: (ii) $h_1(a) = y_1$ and $h_2(a) = \underline{\quad 5 \quad}$. Combining (i) and (ii) we see that $g \circ h_1 = g \circ h_2$. Using property B we get $\underline{\quad 6 \quad}$, proving C.

Suppose C is true. Define f as follows. For a given z in Z, if $z = g(y)$ for some y, then such y is $\underline{\quad 7 \quad}$ element of Y and we define $f(z) = \underline{\quad 8 \quad}$. If $z \neq g(y)$ for any y, then define $f(z)$ to be $\underline{\quad 9 \quad}$ element of Y. It is easy to see that $f \circ g = id_Y$, proving A.

Options for the blanks

A. h_1
B. h_2
C. g
D. f
E. a
F. y
G. a unique
H. an arbitrary
I. z
J. $y_1 = y_2$
K. $y_1 \neq y_2$
L. y_1
M. $g(y_1) = g(y_2)$
N. $g(y_1) \neq g(y_2)$
O. y_2
P. commutativity
Q. associativity
R. left
S. right

Questions 15 and 16 on the next page

Questions 15 and 16

(15) Type a sequence of 6 letters indicating the correct options to fill in the numbered blanks 1 to 6. For example, **BRCBEJ** is in the correct format. [3 points]

(16) Type a sequence of 4 characters as follows: 3 letters indicating the correct options to fill in the numbered blanks 7 to 9, followed by a single number from 1 to 4 that you should choose as follows. If the set Y is allowed to be empty, find the first paragraph in the above reasoning that becomes invalid and write its number. If all three paragraphs stay fully valid, write 4. E.g., **QJE2** is in the correct format. [3 points]

Solutions

Proof of equivalence: Suppose property A is true. If $g \circ h_1 = g \circ h_2$, then applying the function f on the left of both sides of this equation gives $h_1 = h_2$ by associativity of function composition, proving B.

Suppose B is true. To prove C, suppose $g(y_1) = g(y_2)$ where y_1 is in Y and y_2 is in Y. Take $X = \{a\}$, a set with a single element a. Define functions h_1, h_2 from this set X to the set Y as follows: (ii) $h_1(a) = y_1$ and $h_2(a) = y_2$. Combining (i) and (ii) we see that $g \circ h_1 = g \circ h_2$. Using property B we get $y_1 = y_2$, proving C.

Suppose C is true. Define f as follows. For a given z in Z, if $z = g(y)$ for some y, then such y is a unique element of Y and we define $f(z) = y$. If $z \neq g(y)$ for any y, then define $f(z)$ to be an arbitrary element of Y. It is easy to see that $f \circ g = id_Y$, proving A.

If Y is empty, the penultimate sentence (containing blank 9) becomes invalid, because there is no element in Y that can be chosen. Every sentence before that remains valid. Working this out fully is a good way to understand what it means to be "vacuously true". For example, see that when Y is the empty set, property C is true (for any set Z and the unique function $g : \phi \to Z$), B is (vacuously) true, but A is not unless Z is also empty.

Answers: (15) DRQMOJ (16) GFH3 (partial credit for getting GFH correct)

Answer Key for part A

(2) 5,33

(3) 3,11

(4) 15,68

(5) 47

(6) 300

(7) 3

(8) 17,13,20

(9) 1,2,4

(10) 0.347

(11) CIGNGQJD

(12) GBEBDKGG

(13) FYKAI

(14) 128,8,124,64

(15) DRQMOJ

(16) GFH3

Part B Solutions start on the next page

B1. [**14 points**] Let $r_1, \ldots, r_7$ be *positive* numbers less than $\frac{1}{2026}$. Define a polynomial $p(x)$ by

$$p(x) = \prod_{i=1}^{7}(x + r_i) = a_0x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7.$$

Show carefully that

$$\sum_{i=1}^{7} \tan^{-1} r_i = \tan^{-1}\left(\frac{a_1 - a_3 + a_5 - a_7}{a_0 - a_2 + a_4 - a_6}\right).$$

Solution. Let $\tan^{-1} r_i = \theta_i$ for $i = 1, \ldots, 7$. We will show by induction on n that

$$(*) \qquad \tan(\theta_1 + \cdots + \theta_n) = \frac{e_1(n) - e_3(n) + e_5(n) - \cdots}{e_0(n) - e_2(n) + e_4(n) - \cdots}, \text{ where}$$

$e_k(n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} r_{i_1} \cdots r_{i_k}$ is the "elementary symmetric function" in $r_1, \ldots, r_n$. So $e_0(n) = 1$ is the 0-fold product, $e_1(n) = \sum_{1 \le i \le n} r_i = r_1 + \cdots + r_n$,
$e_2(n) = \sum_{1 \le i < j \le n} r_i r_j = r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n$, and so on till $e_n(n) = r_1r_2 \cdots r_n$.

Note that $a_k = e_k(7)$ by Vieta's formulas (or expand $p(x) = \prod_{i=1}^{7}(x + r_i)$ and collect terms). The proof of $(*)$ is a matter of careful bookkeeping using $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$, which gives the case $n = 2$ (base case $n = 1$ is immediate). Then $n = 3$ is a short calculation:

$$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{\frac{r_1+r_2}{1-r_1r_2} + r_3}{1 - \frac{r_1+r_2}{1-r_1r_2} r_3} = \frac{e_1(3) - e_3(3)}{1 - e_2(3)}.$$

In general we proceed similarly. Use $e_k(n+1) = e_k(n) + e_{k-1}(n)r_{n+1}$ to simplify the RHS of

$$\tan(\theta_1 + \cdots + \theta_n + \theta_{n+1}) = \frac{\frac{e_1(n)-e_3(n)+e_5(n)-\cdots}{e_0(n)-e_2(n)+e_4(n)-\cdots} + r_{n+1}}{1 - \left(\frac{e_1(n)-e_3(n)+e_5(n)-\cdots}{e_0(n)-e_2(n)+e_4(n)-\cdots}\right) r_{n+1}}.$$

To get the result in the question, take $n = 7$, apply $\tan^{-1}$ to both sides of the proved equality $(*)$, and see that $\tan^{-1}(\tan(\theta_1 + \cdots + \theta_7)) = \theta_1 + \cdots + \theta_7$. Reason: as $0 < r_i < \frac{1}{2026}$, we have $0 < \theta_i = \tan^{-1} r_i < r_i < \frac{1}{2026}$, so $0 < \theta_1 + \cdots + \theta_7 < \frac{\pi}{2}$. (Note: the word *positive* was missing in the exam wording, making the preceding "$0 <$" unjustified. The question will be graded taking this into account.)

Solution 2. Use complex numbers. Find $p(i)$ in two ways. Multiply by i for convenience.

$$ip(i) = i \cdot i^7[(a_0 - a_2 + a_4 - a_6) + i(-a_1 + a_3 - a_5 + a_7)] = (a_0 - a_2 + a_4 - a_6) + i(-a_1 + a_3 - a_5 + a_7).$$

But we also have $ip(i) = i\prod_{j=1}^{7}(i + \tan(\theta_j)) = i\prod_{j=1}^{7}\frac{i\cos(\theta_j)+\sin(\theta_j)}{\cos(\theta_j)} = \frac{i^8\prod_{j=1}^{7}\big(\cos(\theta_j)-i\sin(\theta_j)\big)}{\prod_{j=1}^{7}\cos(\theta_j)}$
$=$ (a real number) $\prod_{j=1}^{7} e^{-i\theta_j} =$ (a real number) $e^{-i\sum_{j=1}^{7}\theta_j}$.

Let $ip(i) = re^{i\alpha}$. We have $-\tan(\alpha) = (a_1 - a_3 + a_5 - a_7)/(a_0 - a_2 + a_4 - a_6)$ by the first calculation and $-\tan(\alpha) = \tan(\sum_{j=1}^{7}\theta_j)$ by the second calculation. Apply $\tan^{-1}$. As in the previous solution, θ_j are positive angles smaller than $1/2026$, so their sum is less than $\pi/2$.

B2. [**14 points**] For a function f suppose that f', f'' and f''' exist.

(a) Must the following limit exist? Justify your answer and if the limit exists, calculate it.

$$\lim_{n\to\infty}\left(n\int_0^1 x^n f(x)dx\right).$$

(b) Suppose the following limit is a real number a. Calculate $f(1)$.

$$\lim_{n\to\infty}\left(n^2\int_0^1 x^n f(x)dx\right).$$

Solution. (a) The answer is $f(1)$. To see this, let $u = f(x)$, $dv = x^n dx$. Integrate by parts:

$$n\int_0^1 x^n f(x)dx = \frac{n}{n+1}\left(x^{n+1}f(x)\right)\Big|_0^1 - \frac{n}{n+1}\int_0^1 x^{n+1}f'(x)dx.$$

When we take the limit as $n \to \infty$, the first term on the RHS gives $f(1)$. The second term goes to 0 for the following reason: f' is continuous (as f'' exists), so by the extreme value theorem, on the closed interval $[0, 1]$ we have $|f'(x)| <$ some constant M. See graphically what happens to the factor x^{n+1} over [0,1] as $n \to \infty$. Formally we argue as follows:

$$\left|\int_0^1 x^{n+1}f'(x)dx\right| \le \int_0^1 x^{n+1}|f'(x)|dx \le M\int_0^1 x^{n+1}dx = M\frac{x^{n+2}}{n+2}\Big|_0^1 \to 0 \text{ as } n \to \infty.$$

(b) The answer is 0. To see this, we again integrate by parts with $u = f(x)$, $dv = x^n dx$.

$$n^2\int_0^1 x^n f(x)dx = \frac{n^2}{n+1}\left(x^{n+1}f(x)\right)\Big|_0^1 - \frac{n^2}{n+1}\int_0^1 x^{n+1}f'(x)dx.$$

We examine each term on the RHS. For the integral in the second term, apply the same logic as in part (a), but now with f' in place of f (and use that $\frac{n^2}{n+1}$ behaves like n in the limit). So the second term has a finite limit, namely $-f'(1)$. Note that now extreme value theorem is applied to f'', which is continuous because f''' exists.

Thus for the RHS to have a finite limit, the first term must also have a finite limit. The first term evaluates to $\frac{n^2}{n+1}f(1)$. As $n \to \infty$, the limit of this term is 0 if $f(1) = 0$ and $\pm\infty$ otherwise. So we must have $f(1) = 0$ (and $a = -f'(1)$, but that was not asked).

Note: for integration by parts to be valid, it is sufficient for u and v to be continuously differentiable, which is the case here.

B3. **[14 points]** Let g be a differentiable function with $g(0) = 0$ and $g(1) = 1$. Suppose that for any real number r, you have a black box that allows you to find all solutions of $g(x) = r$ as well as of $g'(x) = r$. For any positive integer n, state with proof a procedure to find distinct $c_1, c_2, \ldots, c_n$ in the interval (0,1) such that

$$\sum_{i=1}^{n} \frac{1}{g'(c_i)} = n.$$

Hint: For $n = 2$, first choose a suitable point P on the graph of g. Consider the two line segments connecting P with points (0,0) and (1,1) respectively.

Solution. By repeated use of the intermediate value theorem (and the black box), get a sequence of points $x_0 = 0, x_n = 1, x_1, x_2, \ldots x_{n-1}$ with $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that $g(x_i) = i/n$. This works for $i = 0$ and $i = n$ too as $g(0) = 0$ and $g(1) = 1$.

Then use the mean value theorem and the black box to get c_i between x_{i-1} and x_i. We have

$$g'(c_i) = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} = \frac{\frac{i}{n} - \frac{i-1}{n}}{x_i - x_{i-1}} = \frac{1}{n(x_i - x_{i-1})}, \text{ so}$$

$$\sum_{i=1}^{n} \frac{1}{g'(c_i)} = \sum_{i=1}^{n} (n(x_i - x_{i-1})) = n \sum_{i=1}^{n} (x_i - x_{i-1}) = n(1 - 0) = n.$$

(For $n = 2$, the hint suggests choosing a point, say $P = (a, g(a))$. Let $b = g(a)$. Use of MVT suggests itself in view of the problem statement/hint, giving $0 < c_1 < c_2 < 1$ with slopes $g'(c_1) = b/a$ and $g'(c_2) = (1-b)/(1-a)$. Now $\frac{1}{g'(c_1)} + \frac{1}{g'(c_2)} = \frac{a}{b} + \frac{1-a}{1-b}$. A simple way for this to be 2 (as desired) is to take $b = g(a) = 1/2$ so that both denominators become 1/2.)

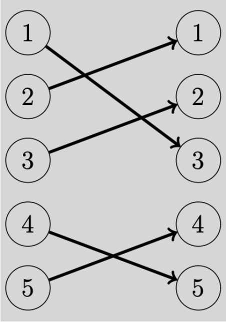
B4. **[14 points]** Let a be a positive integer. Suppose that $x^{2026} - a = g(x)h(x)$, where $g(x)$ and $h(x)$ are nonconstant polynomials with *integer coefficients*. Show that $a = b^2$ for some integer b or $a = c^{1013}$ for some integer c (or both). Possible hint: what are the complex roots of $x^{2026} - a$?

Solution. We prove the result more generally by replacing 1013 by any odd prime p. The $2p$ complex roots of $f(x) = x^{2p} - a$ lie on the circle of radius $a^{1/2p}$ with center at 0.

Suppose $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are nonconstant polynomials with *integer coefficients*. The complex roots of $g(x)$ have to be among the complex roots of $f(x)$. (Why?) Let r be the degree of $g(x)$, so $1 \leq r < 2p$. The constant term of $g(x)$ has absolute value $a^{r/2p}$. (Why?) Since this term is an integer, it has to be real and hence must be $\pm a^{r/2p}$. (Why?) Say $a^{r/2p} =$ the integer m, so we have the integer equation $a^r = m^{2p}$. Let $d = \gcd(r, 2p)$. Then $d = 1, 2$, or p. We examine the three cases one by one.

(i) If $d = p$, then r has to be p, so $a^p = m^{2p}$, giving $a = m^2$. (ii) If $d = 2$, $r = 2s$ is an even number with $s < p$. Matching the exponents in the prime factorisations on either side of $a^r = m^{2p}$ gives that every exponent of a prime factor in factorization of a must be divisible by p. (See this carefully.) It follows that $a = c^p$ for some c. (iii) If $d = 1$, i.e., when r and $2p$ are coprime, the same argument gives that $a = c^{2p}$, which is both a square and a p-th power.

B5. **[24 points]** Consider the set of all permutations of the set $[n] = \{1, 2, \ldots, n\}$, i.e., the set of all bijective functions from $[n]$ to $[n]$.



It may be useful to think about a permutation f in two ways.

(1) As the rearrangement $f(1), f(2), \ldots, f(n)$ of $[n]$.

(2) As a function represented pictorially: draw two vertical columns of dots, with dots in each column labeled $1, 2, \ldots, n$ in that order from top to bottom. Then draw an arrow from each i to $f(i)$. For example, the picture to the right is such a representation of the permutation 3 1 2 5 4.

Definition 1: A permutation f is *reducible* if there is an integer k with $1 \leq k < n$ such that $f(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$. This means that in the picture representing the function, a horizontal line drawn in the gap between k and $k+1$ is not crossed by any arrow. If f is not reducible, it is *irreducible*. The permutation in the example is reducible by taking $k = 3$.

Definition 2: An *inversion* in a permutation f is a pair (i, j) of indices in $[n]$ such that $i < j$ and $f(i) > f(j)$. In the picture representing f, an inversion (i, j) is seen as two arrows, from i to $f(i)$ and from j to $f(j)$, that intersect. In the example $(1, 3)$ is an inversion as $f(1) > f(3)$. But $(2, 3)$ is not as $f(2) < f(3)$. Let $I(f) =$ the number of inversions in f.

Solve the following independent questions. Part (a) may be considerably simpler.

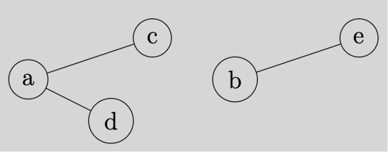
(a) Determine, with proof, the *maximum* possible value of $I(f)$ for a *reducible* permutation f. Construct a reducible permutation that achieves this number.

(b) Determine, with proof, the *minimum* possible value of $I(f)$ for an *irreducible* permutation f. Construct an irreducible permutation that achieves this number.

Here are two possible methods for part (b). Label your answer as Method 1, Method 2 or (your own) Method 3.

Method 1: Define *gap crossing number* of f as $C(f) = |f(1)-1| + |f(2)-2| + \cdots + |f(n)-n|$. Now consider, if possible, an adjacent inversion in f, i.e., $(i, i+1)$ with $f(i) > f(i+1)$. How does swapping $f(i)$ and $f(i+1)$ affect $I(f)$ and $C(f)$? Can you relate $I(f)$ and $C(f)$?

Method 2: This uses the language of graphs, which is easy to understand by an example. The graph G in the picture to the right has vertices a, b, c, d, e and edges $\{a, c\}$, $\{a, d\}$, and $\{b, e\}$. This G is disconnected, with two connected components, namely $\{a, c, d\}$ and $\{b, e\}$. What is the minimum number of edges in a connected graph (i.e., a graph having only one component)?



Define the *inversion graph* $G(f)$ of a permutation f as having vertices $1, 2, \ldots, n$ with an edge $\{i, j\}$ precisely when (i, j) is an inversion in f. Prove that if a and b are vertices in a connected component of $G(f)$ with $a < b$, then any integer c such that $a < c < b$ must also belong to the same component. Deduce that if $G(f)$ is disconnected then f is reducible.

Solution. (a) By definition $I(f) \leq \binom{n}{2}$,with equality only for the reversing permutation, the only one for which every pair gives an inversion. For f reducible, say with $f([k]) = [k]$, clearly the maximum value of $I(f)$ is obtained when f reverses $1, 2, \ldots, k$ and also reverses $k+1, \ldots, n$. For this f, we have $I(f) = \binom{k}{2} + \binom{n-k}{2}$. This is a quadratic in k with a positive coefficient for k^2, so its maximum is obtained at an endpoint of the interval $[1, n-1]$. (Or see by working out the formula.) By symmetry either $k = 1$ or $k = n-1$ gives maximum $I(f) = \binom{n-1}{2}$.

(b) **Method 1.** Show that $I(f) \geq C(f)/2$ by induction on $I(f)$. Reason: both numbers are 0 for the identity permutation. Otherwise take an out-of-order adjacent pair, i.e., one with $f(i) > f(i+1)$. Swapping such a pair reduces $I(f)$ by exactly 1. (Why?) And the same operation either reduces $C(f)$ by 2 (this happens precisely when $f(i) > i$ and $f(i+1) < i+1$) or keeps $C(f)$ the same (this happens otherwise, precisely when either $f(i+1) < f(i) \leq i$ or when $f(i) > f(i+1) \geq i+1$).

Now for an irreducible f, the gap between each k and $k+1$ has to be crossed by at least two arrows, one "pointing down" ($i \leq k$ and $f(i) > k$) and one "pointing up" ($j > k$ and $f(j) \leq k$). Each such crossing contributes at least 1 to the total displacement sum $C(f)$. So for an irreducible f, we have $C(f) \geq 2(n-1)$ and hence $I(f) \geq C(f)/2 \geq n-1$.

To construct an irreducible f with fewest possible inversions, we try to keep as many pairs in order as possible while ensuring that every $[k]$ "spills over" under f by having an arrow go out to something bigger than k. A simpleminded scheme is to have $f(1) = 2$, then $f(2) = 3$ to avoid inversion *and* to continue spillover, and so on till $f(n-1) = n$. We are forced to end by defining $f(n) = 1$. See that $2, 3, 4, \ldots, n, 1$ is irreducible with exactly $n-1$ inversions.

Method 2. Proceed by contradiction to prove the claim. For the inversion graph of f, let A be the connected component containing vertices a, b with $a < b$. Suppose $c \notin A$, so vertex c has no edge to *any* vertex v in A. This enforces the following.

(i) If $v < c$, we must have $f(v) < f(c)$. (ii) If $v > c$, we must have $f(v) > f(c)$.

Partition the vertices of A into two sets: $A_{left} = \{v \in A \mid v < c\}$, $A_{right} = \{v \in A \mid v > c\}$. Note that $a \in A_{left}$ and $b \in A_{right}$, so neither set is empty. For any $x \in A_{left}$ and $y \in A_{right}$, the inequalities give us $f(x) < f(c) < f(y)$ and therefore $f(x) < f(y)$. Since $x < y$ and $f(x) < f(y)$, there can be no edge between x and y. This means there are no edges between A_{left} and A_{right}. But $A_{left} \cup A_{right} = A$, and A is supposed to be connected. This is a contradiction. Thus, $c \in A$. Thus any connected component of an inversion graph consists of a consecutive block of numbers in $[n]$.

It follows that the inversion graph of an irreducible f must be connected. If not, take the first connected component $C_1 = \{1, \ldots, k\}$ and see that $f(C_1) = C_1$, forcing f to be reducible. Reason: we have $x < y$ for any $x \in C_1$ and $y \notin C_1$. Then we also have $f(x) < f(y)$ as there are no edges between C_1 and the rest of the graph. This means that *every* value assigned to the first k elements is strictly smaller than *every* value assigned to the remaining $n-k$ elements. The only way this is possible for a bijection on $\{1, \ldots, n\}$ is if $f(C_1) = C_1$. (Converse is true and easier: connected $G(f)$ means f is irreducible. But not needed here.)

A connected graph with n vertices must have at least $n-1$ edges. (Why?) So the inversion graph of an irreducible f, being connected, must have at least $n-1$ edges, i.e., f must have at least $n-1$ inversions. As before, $I(f) = n-1$ is achieved by $2, 3, 4, \ldots, n, 1$. (And similarly also by $n, 1, 2, \ldots, n-1$. Question: are there any other irreducible f with $I(f) = n-1$?)