CMI BSc entrance exam on May 24, 2025

Unless specified otherwise, in this exam all numbers are real and "function" means a function whose codomain as well as domain is the set of all real numbers or an implied subset. You may use the following information if you find it relevant.

$$2025 = 3^4 \times 5^2 \qquad \tan(\theta) = \frac{\sin(2\theta)}{1+\cos(2\theta)}$$

Instructions for Part A

- **Part A is worth 40 points.** It has 17 questions, which are numbered 2 to 18 for technical reasons. *There is no negative marking.*
- For each of the Part A questions, **type in your answer** as directed.
  - In questions where multiple choices are given, type the label(s) of the correct option(s) from the given list. There are no True/False questions.
  - For numerical answers, unless specified otherwise, enter the closest integer. E.g., enter 3 if the answer is any of $e, 3, \pi$, but enter 4 for answer 3.5.
  - In remaining cases, the format of the answer is explained with the question and an example is given in **blue**. *Read the instructions carefully.*
- **Part A will be used for screening.** Part B is assured to be graded if you meet *any one* of the following two conditions. (i) You score at least 24 in part A. (ii) You are among the top 400 students in part A. Thus part B will be graded for at least 400 students, more if enough students score at least 24 in part A.

Instructions for Part B

- **Part B has 6 problems worth a total of 80 points.** Solve each part B problem on the designated pages in the answer booklet.
- **Clearly explain your entire reasoning**. No credit will be given without correct reasoning. You may solve a later part of a problem by assuming some previous part(s), even if you could not do the earlier part(s).
- You are advised to spend at least 2 hours on part B.

Part A questions

Question numbering deliberately starts at 2.

Information for questions (2) to (5)

Let $S = \{1, 2, \ldots, 100\}$. Randomly pick an element x from S, every element being equally probable. Let S_1 be the set of all elements in S *less than or equal to* x and let S_2 be the set of all remaining elements of S, i.e., elements *strictly bigger than* x. Answer questions (2) to (5) as per the given instruction.

Instruction for (2) to (5)

If the probability is $p\%$, then your answer should be the integer closest to p. E.g., for probability $\frac{1}{3} = 33.33\ldots\%$, you should type 33 as your answer. For probability $\frac{2}{3}$ you should type 67 as your answer.

Questions

(2) What is the probability that 50 belongs to a set of size exactly 60? [1 point]

(3) What is the probability that 20 belongs to S_1 and 60 belongs to S_2? [1 point]

(4) What is the probability that 20 and 60 are both in S_1 or they are both in S_2? [1 point]

(5) What is the probability that the product of sizes $|S_1||S_2| > 900$? [2 points]

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(6) Let $P = (a, b, c)$ be the point on the plane $3x + 5y - 7z = 9$ that is closest to the point $(8, 12, -13)$. Write the integer that is closest to $a^2 + b^2 + c^2$. [3 points]

(7) Write the value of the following number correct to two decimal places. E.g., for e type 2.72 and for 3 type 3.00. If the limit does not exist, type DNE as your answer. [4 points]

$$\lim_{t \to 0^+} \left(\lim_{x \to 0^+} \frac{\ln(1 + tx)}{\int_0^x \sqrt{e^x - \cos(t)\cos(x)}\, dx} \right)$$

Information for questions (8) and (9)

A and B are points on a paper with $AB = 10$. A fly walks on the paper from A to B in such a way that for any third point P on the path, $\angle APB = \alpha$, where $\alpha =$ the angle opposite to side 4 in a triangle with sides $3, 4, 5$.

Questions

(8) The path traced out by the fly is part of _____ . Fill in the blank with the number of one of the following options. [1 point]

1. a straight line
2. a circle
3. an ellipse
4. a parabola
5. a hyperbola
6. none of the options numbered 1 to 5

(9) For each point P along the path, consider $\triangle APB$. Write the maximum possible area of such a triangle. [3 points]

Information for questions (10) and (11)

Let C be the curve defined by $y = x^2$ where $0 \leq x \leq 2$. Let $P = (9, -3)$.

Questions

(10) The x-coordinate of a point on C closest to P satisfies $ax^3 + bx = c$ where a, b, c are integers with $gcd(a, b, c) = 1$. Write the values of a, b, c separated by commas with no gaps. E.g., for $x^3 + 2x = 3$, the answer would be 1,2,3. [2 points]

(11) Let $d =$ the minimum distance of P from a point on C. Let $D =$ the maximum distance of P from a point on C. Write two integers separated by only a comma: the integer closest to d^2, the integer closest to D^2. E.g., 20,25 is an answer in the correct format. [3 points]

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(12) Count the number of *ordered* tuples of *integers* (a, n_1, n_2, n_3, n_4) such that all three conditions below are satisfied. [4 points]

- $a > 0$.
- $n_i \geq -1$ for each i.
- $a^2 + n_1 + n_2 + n_3 + n_4 = 5$.

Information for question (13)

Claim: Suppose a, b are distinct roots of a polynomial $p(x)$. Then $p(x)$ is a multiple of $(x-a)(x-b)$.

Proof from first principles: Use long division to get $p(x) = (x-a)q(x) + r$ where __1__ is some constant. Substituting $x =$ __2__, we get $r =$ __3__, so __4__ is a multiple of $(x-a)$. Now substituting $x =$ __5__ we get that __6__ is a root of __7__. Finally, again apply the logic in the first two sentences, now to $q(x)$. This shows that __8__ is a multiple of __9__, completing the proof.

Options for the blanks

A. a B. b C. r

D. 0 E. $q(a)$ F. $q(b)$

G. $q(r)$ H. $p(x)$ I. $q(x)$

J. $(x-a)$ K. $(x-b)$ L. $(x-r)$

(13) Complete the given proof by writing a sequence of nine letters indicating the correct options to fill in the numbered blanks 1 to 9. Do not use any spaces, full stop or any other punctuation. E.g., **ABACDIJKB** is in the correct format. [3 points]

Information for question (14)

Logic similar to Question (13) works for a polynomial $P(x, y)$ in two variables: if after substituting $x = f(y)$ one gets $P(f(y), y) = 0$, then $P(x, y) = (x - f(y))Q(x, y)$. Similarly if $P(g(y), y) = 0$ as well, then an analogous conclusion holds. Assume this.

Now let $A(x, y)$ be a *symmetric* polynomial, meaning $A(x, y) = A(y, x)$. Let $\omega = e^{2\pi i/3}$. Suppose that substituting $x = \omega y$ gives $A(\omega y, y) = 0$. Then $A(x, y)$ is a multiple of __1__. Therefore it is a multiple of __2__ too. Therefore $A(x, y)$ must be a multiple of the following polynomial with integer coefficients: __3__.

Options for the blanks

A. $(x - \omega y)$ B. $(y - \omega x)$

C. $(x + \omega y)$ D. $(y + \omega x)$

E. $(x + y)$ F. $x^2 - 2xy + y^2$

G. $x^2 - xy + y^2$ H. $x^2 + y^2$

I. $x^2 + xy + y^2$ J. $x^2 + 2xy + y^2$

(14) Complete the reasoning about $A(x, y)$ by writing a sequence of three letters indicating the correct options to fill in the numbered blanks 1 to 3. E.g., **ABC** is in the correct format. [2 points]

Information for question (15)

Claim: $x = y = z = 0$ is the only integer solution to $x^2 + y^2 = 105z^2$.

Proof: Consider an integer solution (x, y, z) with the __1__ possible value of $|x| + |y| + |z|$.

When any perfect square is divided by 3 the possible remainders are 0 and 1. Therefore, when each term on the left hand side $x^2 + y^2$ is divided by 3, the possible combinations of remainders are $0 + 0, 0 + 1, 1 + 0$, and $1 + 1$.

It follows that __2__ must be divisible by __3__. Therefore $x^2 + y^2$ is divisible by __4__. Therefore __5__ is divisible by __6__ because 105 is divisible by __7__ but not by __8__. Now observe that __9__ x, y and z by __10__ still gives a solution of the given equation. Unless $x = y = z = 0$, this contradicts the first sentence of the proof.

Options for the blanks

A. largest	B. smallest
C. x	D. y
E. z	F. x or y
G. both x and y	H. x or y or z
I. all of x, y, z	J. dividing
K. multiplying	An integer from 0 to 9

(15) Fill each blank with the letter of an option OR an integer between 0 to 9. For example **K9AC6JK1BB** is an answer in the correct format. [4 points]

Information for question (16)

Can the argument in Question (15) work with the exact same logic to prove the same claim for the same integer equation $x^2 + y^2 = 105z^2$, if we divide by a number other than 3? You are asked to test this for 5 and 7 and answer as explained in the question.

(16) Write your answer as a single letter from options A,B,C,D followed by listing in increasing order the letter of every equality that is NOT encountered while deciding the correct option from A to D. For example, **AGI** is an answer in the correct format. [2 points]

Options for (16)

A. Same argument works for 5 as well as 7
B. Same argument works for 5 but not 7
C. Same argument works for 7 but not 5
D. Same argument works for neither 5 nor 7

E. $0 + 1 = 1$	F. $1 + 1 = 2$	G. $1 + 2 = 3$	H. $1 + 4 = 5$
I. $2 + 2 = 4$	J. $3 + 3 = 6$	K. $4 + 4 = 8$	

Information for questions (17) and (18)

Suppose a function $f(x)$ has domain $\mathbb{R}$ and satisfies the following three conditions.

(A) f is differentiable.
(B) f is increasing.
(C) $0 < f(x) \leq 1$ for each x.

Suppose $\lim_{x\to\infty} f'(x) =$ a real number L. Complete the following proof showing $L = 0$.

Proof: By the given condition __1__, the value of L must be __2__ __3__. If L is nonzero, we can choose a large enough N such that for x__4__N, the value of $f'(x)$ is $\geq$ __5__. Therefore by the following theorem __6__, for any $x \geq$ __7__, given condition __8__ will be violated, giving a contradiction.

Options for the blanks

A. (A)	B. (B)	C. (C)
D. $=$	E. $\geq$	F. $\leq$
G. 0	H. 1/2	I. 1
J. $L/2$	K. $2L$	L. L
M. $N + \frac{L}{2}$	N. $N + L$	O. $N + 2L$
P. $N + \frac{2}{L}$	Q. $N + \frac{1}{L}$	R. $N + \frac{1}{2L}$

S. Mean value theorem

T. Extreme value theorem

U. Intermediate value theorem

V. Fundamental theorem of calculus

Questions

(17) Write a sequence of 5 letters indicating the correct options to fill in the numbered blanks 1 to 5. For example, **BACDE** is in the correct format. [2 points]

(18) Write a sequence of 3 letters indicating the correct options to fill in the numbered blanks 6 to 8. E.g., **WJE** is in the correct format. [2 points]

Part B Problems for BSc entrance exam on May 24, 2025

B1. [**12 points**] Suppose five complex numbers z_1, z_2, z_3, z_4, z_5 form the vertices of a regular pentagon that is inscribed in a circle of radius 2 with center at $c = 6 + 8i$.

(a) Find all possible values of $S = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2$. State a value of z_1 maximizing $|S|$.

(b) Find all possible values of $P = z_1 z_2 z_3 z_4 z_5$. State a value of z_1 minimizing $|P|$.

B2. [**12 points**] Consider the following functions from $\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$ to itself. Let R_α be counterclockwise rotation by angle α and let F_α = reflection in the line that makes counterclockwise angle α with the X-axis. E.g., $R_{90^\circ}(1,0) = (0,1)$ and $F_{90^\circ}(1,0) = (-1,0)$.

(a) Evaluate $R_\alpha(r\cos\theta, r\sin\theta)$ and $F_\alpha(r\cos\theta, r\sin\theta)$, where $r \geq 0$ and θ is any angle.

(b) Geometrically describe the composition of functions $F_\alpha \circ F_0$. (Note that F_0 = reflection in the X-axis.) You may use any valid method as long as you explain clearly.

(c) For any angles α and β, geometrically describe $F_\alpha \circ F_\beta$ in one crisp sentence. You may appeal to (b) if you justify why and how your work in (b) can be used here. Find $G(P)$ for $G = (F_{20^\circ} \circ F_{25^\circ})^9 = (F_{20^\circ} \circ F_{25^\circ})$ composed with itself 9 times and P = the point $(20, 25)$.

B3. [**12 points**] A particle starts at (0,0) and travels in the first quadrant along a straight line. It maintains the slope AND stays in the square bounded by the lines $x = 0, x = 1, y = 0$ and $y = 1$ as follows. Whenever it reaches the boundary of this square it magically jumps 1 unit to the left or down or both as applicable. In other words, for $a < 1$, it jumps to $(a, 0)$ upon reaching $(a, 1)$, it jumps to $(0, a)$ upon reaching $(1, a)$, and it jumps to (0,0) if it reaches (1,1). If the particle ever reaches a previously visited point, it stops.

For example, $(0,0) \to (1/2, 1) \xrightarrow{jump} (1/2, 0) \to (1,1) \xrightarrow{jump} (0,0)$ is a trajectory of length $\sqrt{5}$. (Jumps don't count for length.) Another way to visualize this trajectory is to let the particle continue across the boundary $y = 1$ and interpret what happens.

(a) What are all the possible stopping points for finite trajectories?

(b) If the particle starts at the angle 30° with respect to the X-axis, show that it never stops.

(c) Find the two smallest possible *integer* lengths among all finite trajectories.

B4. [**12 points**] The domain of f is the set of *positive integers* and $f(xy) = f(x) + f(y)$ for all x, y. Answer the *independent* questions below. (Data from (a) are not valid for the rest.)

(a) Suppose $f(2025) = 0, f(20) = 10$ and $f(25) = 20$. What is the smallest n for which $f(n)$ is not uniquely determined? Write values of $f(x)$ for each positive integer $x < n$.

(b) Is there such a function f for which $f(x) = 0$ for all positive integers $x < 2025^{2025}$ but f is not identically 0? Show how to define such f or show that it is not possible.

(c) The domain of a function g is the set of *positive rational* numbers, *codomain the set of integers**, and $g(xy) = g(x) + g(y)$ for all positive rational x, y. Suppose $g(a) = 24$, $g(b) = 2025, g(c) = 10^{2025}$ for some rational numbers a, b, c. Show that there are infinitely many rational numbers r such that $g(r) = 1$. (*See solutions for relevance of the codomain.)

B5. **[16 points]** Solve the following. Part (b) can be done independently and may be easier.

(a) Construct a function f with domain $\mathbb{R}$ such that f is differentiable, weakly increasing, bounded, and $\lim_{x\to\infty} f'(x)$ does not exist. (*Weakly increasing* means $f(a) \leq f(b)$ for $a < b$. *Bounded* means there are constants m, M such that for every real x, we have $m < f(x) < M$.)

Possible hints: What kind of function should $f'(x)$ be to satisfy the requirements? Thinking in terms of pictures may help. Is there a way to construct a function whose derivative is a desired function? If needed, you may take the domain of f to be $[0, \infty)$ instead.

(b) Construct a function g with domain $\mathbb{R}$ such that g is differentiable, *strictly* increasing, bounded, $\lim_{x\to-\infty} g(x) = 0$, $\lim_{x\to\infty} g'(x)$ does not exist, and $\lim_{x\to-\infty} g'(x)$ does not exist. (*Strictly increasing* means $g(a) < g(b)$ for $a < b$.)

Possible hint: You may use your answer to part (a) and adjust as necessary. Even if you did not do part (a), you may take as given a function f with domain $[0, \infty)$ and the required properties in (a). Then show with clear explanation how to build g in terms of f.

B6. **[16 points]** $a_1, a_2, a_3, \ldots, a_n$ is a sequence of *distinct* numbers, each written on a card. Take the cards one by one in order. Place them into stacks subject to this rule: *one can only place a smaller number on top of a larger number or one can start a new stack with a card.*

Here is a simple greedy strategy. Make stacks along a line and *place the incoming card on top of the leftmost stack possible.* So if the incoming card cannot be placed on any existing stack, then we use that card to start a new stack to the right of all current stacks.

For example, with the sequence 3 7 2 5 6 4 9 8 we get the four stacks:

$$\begin{array}{cccc} & 4 & & \\ 2 & 5 & & 8 \\ 3 & 7 & 6 & 9 \end{array}$$

(a) Show that under the greedy strategy, the top numbers on the stacks increase from left to right, e.g., 2 4 6 8 in the example above.

(b) Show that under the greedy strategy, the number of stacks is the length of a longest possible increasing subsequence of $a_1, a_2, a_3, \ldots, a_n$. A subsequence means $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. The example given above has an increasing subsequence of length 4 (e.g., 3 5 6 8 among others) but none of length 5. It also gives 4 stacks as claimed.

Possible hint: for an entry x in the given sequence, let $\ell(x)$ = the length of a longest increasing subsequence whose last entry is x. What are the values of $\ell(x)$ in the example?

(c) Show that the greedy strategy gives the minimum possible number of stacks.

(d) Find two sequences that give the same end result of stacks after using the greedy strategy.

(e) Show that not every sequence of legal stacks is obtainable by using the greedy strategy. (*Legal stack* means numbers increase from top to bottom.) Given a sequence of legal stacks along a line, how will you decide if it arises as the result of using the greedy strategy on some sequence of numbers?