Draft solutions for CMI BSc entrance exam on May 24, 2025

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Part A draft solutions (answer key after the solutions)

Information for questions (2) to (5) (Question numbering deliberately starts at 2.)

Let $S = \{1, 2, \ldots, 100\}$. Randomly pick an element x from S, every element being equally probable. Let S_1 be the set of all elements in S *less than or equal to* x and let S_2 be the set of all remaining elements of S, i.e., elements *strictly bigger than* x. Answer questions (2) to (5) as per the given instruction.

Instruction for (2) to (5)

If the probability is $p\%$, then your answer should be the integer closest to p. E.g., for probability $\frac{1}{3} = 33.33\ldots\%$, you should type 33 as your answer. For probability $\frac{2}{3}$ you should type 67 as your answer.

Questions

(2) What is the probability that 50 belongs to a set of size exactly 60? [1 point]

(3) What is the probability that 20 belongs to S_1 and 60 belongs to S_2? [1 point]

(4) What is the probability that 20 and 60 are both in S_1 or they are both in S_2? [1 point]

(5) What is the probability that the product of sizes $|S_1||S_2| > 900$? [2 points]

Answers

(2) 2% because x must be 60 or 41.

(3) 40% because the condition is equivalent to $20 \leq x \leq 59$.

(4) 60% because this is complementary to (2).

(5) 79% because $x \leq 10$ and $x \geq 90$ are the undesired cases. E.g., see the graph of $x(100-x)$.

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(6) Let $P = (a, b, c)$ be the point on the plane $3x + 5y - 7z = 9$ that is closest to the point $(8, 12, -13)$. Write the integer that is closest to $a^2 + b^2 + c^2$. [3 points]

Answer: (2,2,1) is the closest point so the answer is 9. To get the answer one solves for (a, b, c) and a scalar λ using two equations: (i) $3a + 5b - 7c = 0$ because P is on the plane, and (ii) $(a-8, b-12, c+13) = \lambda(3, 5, -7)$ because the plane by its definition is perpendicular to $(3, 5, -7)$ (to $3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$ if you wish) and also perpendicular to $(a - 8, b - 12, c + 13)$ by virtue of P being closest to $(8, 12, -13)$, thus forcing the proportionalilty.

(7) Write the value of the following number correct to two decimal places. E.g., for e type 2.72 and for 3 type 3.00. If the limit does not exist, type DNE as your answer. [4 points]

$$\lim_{t\to 0^+}\left(\lim_{x\to 0^+}\frac{\ln(1+tx)}{\int_0^x \sqrt{e^x-\cos(t)\cos(x)}\,dx}\right)$$

Answer: $\sqrt{2}$. The outer limit is as $t\to 0^+$, so fix a *positive* value for t in order to first study the inner limit. By substituting $x=0$, the inner limit is "of type 0/0", so to calculate it we may try to use L'Hôpital's rule*. Rushing headlong into it, the derivative (with respect to x) of the numerator is $(\ln(1+tx))' = t/(1+tx)$ and that of the denominator, by the fundamental theorem of calculus, is $\sqrt{e^x-\cos(t)\cos(x)}$. As $x\to 0^+$, the limits of these derivatives are, respectively, t and $\sqrt{1-\cos(t)}$. So we need to *examine* $L=\lim_{t\to 0^+}(t/\sqrt{1-\cos(t)})$. By algebra $L=\lim_{t\to 0^+}(t\sqrt{1+\cos(t)})/\sqrt{1-\cos^2(t)})=\sqrt{2}$. Or use $1-\cos(t)=2\sin^2(t/2)$. Or $\lim_{t\to 0}(t^2/(1-\cos(t))=2$, so $L=\sqrt{2}$. (Why is "taking the square root" valid?) To test the answer numerically, plot $(\ln(1+tx))/(\int_0^x \sqrt{e^x-\cos(t)\cos(x)}\,dx)$ for $t=0.01, 0.001, 0.0001$ in a suitable window, e.g. on desmos. Is the L'Hôpital recipe *logically valid* here? Read on.

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*Note: The useful L'Hôpital's rule, popular in high school calculus, deserves more care in its application than is often accorded to it. For example, consider $\lim_{x\to 0}((x^2\sin(x^{-1}))/\sin x)$. First see that this limit is 0. (How?) Next, as $x\to 0$, we have $\sin x\to 0$ and $x^2\sin(x^{-1})\to 0$ because $\sin(x^{-1})$ is bounded. Here is what happens if you "apply" L'Hôpital's rule:

$$\lim_{x\to 0}\frac{x^2\sin(x^{-1})}{\sin x}\ \stackrel{?}{=}\ \lim_{x\to 0}\left(\frac{2x\sin(x^{-1})}{\cos x}-\frac{\cos(x^{-1})}{\cos x}\right).$$

The RHS is not defined: as $x\to 0$, both $\sin(x^{-1})$ and $\cos(x^{-1})$ oscillate in $[-1,1]$. The first term has limit 0 due to the factor $2x$ but the second has no limit. What went wrong? Nothing. There are other conditions to check before applying the rule. See, e.g., the wikipedia entry for details and more counterexamples. A particular case suffices for question (7): Suppose (i)-(iii) hold: (i) $\lim_{x\to a}N(x)=0=\lim_{x\to a}D(x)$. (ii) For some open interval I containing a and for all $x\in I$ except possibly $x=a$, $N(x)$ and $D(x)$ are differentiable with $D'(x)$ nonzero. (iii) $\lim_{x\to a}(N'(x)/D'(x))$ exists. Call it L. *Then* the rule assures $\lim_{x\to a}(N(x)/D(x))=L$.

Here is a more careful treatment of question (7), showing additionally that the inner limit may be taken to be two sided. Fix $t\in(0,2\pi)$. First, $(1+tx)$ is positive for all $x>(-1/t)$, so the numerator $N(x)=\ln(1+tx)$ is defined for $x\in(-1/t,\infty)$. Next, $e^x-\cos(t)\cos(x)$ is a continuous function of x and (because $0<t<2\pi$), at $x=0$, its value $1-\cos(t)$ is positive. So the integrand $\sqrt{e^x-\cos(t)\cos(x)}$ is defined and >0 for all x in some open interval $I=(-\epsilon,\epsilon)$ containing 0. (Technically, a variable other than x should have been used, e.g., $\int_0^x\sqrt{e^u-\cos(t)\cos(u)}\,du$, but u is dummy anyway.) The integrand is a continuous function of x (or u) on I, so its Riemann integral $\int_0^x$ exists. Thus the numerator $N(x)$ and the denominator $D(x)$ are both defined on I, so it is valid to *consider* the inner limit as $x\to 0$.

Next, $N(x)$ and $D(x)$ are continuous (in fact differentiable) on I, so their limits as $x\to 0$ can be obtained by evaluation at $x=0$. Both limits are 0, so condition (i) for L'Hôpital's rule holds. For $x\in I$, we have $N'(x)=t/(1+tx)$ and $D'(x)=\sqrt{e^x-\cos(t)\cos(x)}>0$, so condition (ii) holds. Condition (iii) holds because, as $x\to 0$, $N'(x)\to t$ and $D'(x)\to\sqrt{1-\cos(t)}>0$. So $\lim_{x\to 0}(N(x)/D(x))=t/\sqrt{1-\cos(t)}$ by L'Hôpital's rule. Now one can continue as in the answer. (Can the outer limit be taken to be two sided? Extend the entire analysis to $t<0$ to show $\lim_{t\to 0^-}(\lim_{x\to 0}(N(x)/D(x)))=-\sqrt{2}$. So the answer is No.)

Information for questions (8) and (9)

A and B are points on a paper with $AB = 10$. A fly walks on the paper from A to B in such a way that for any third point P on the path, $\angle APB = \alpha$, where $\alpha =$ the angle opposite to side 4 in a triangle with sides $3, 4, 5$.

Questions

(8) The path traced out by the fly is part of _____ . Fill in the blank with the number of one of the following options. [1 point]

1. a straight line
2. a circle
3. an ellipse
4. a parabola
5. a hyperbola
6. none of the options numbered 1 to 5

(9) For each point P along the path, consider $\triangle APB$. Write the maximum possible area of such a triangle. [3 points]

Answers

(8) Circle by high school geometry.

(9) 50. Every such triangle has base 10. Area is maximized when the height h is maximum. This happens when the triangle is isosceles. Then $\tan(\alpha/2) = 5/h$. So area $= 25/\tan(\alpha/2)$. From the 3-4-5 right triangle, we have $\sin(\alpha) = 4/5$ and $\cos(\alpha) = 3/5$. The half angle formula gives $\tan(\alpha/2) = (4/5)/(1 + (3/5)) = 1/2$.

Information for questions (10) and (11)

Let C be the curve defined by $y = x^2$ where $0 \leq x \leq 2$. Let $P = (9, -3)$.

Questions

(10) The x-coordinate of a point on C closest to P satisfies $ax^3 + bx = c$ where a, b, c are integers with $gcd(a, b, c) = 1$. Write the values of a, b, c separated by commas with no gaps. E.g., for $x^3 + 2x = 3$, the answer would be 1,2,3. [2 points]

(11) Let $d =$ the minimum distance of P from a point on C. Let $D =$ the maximum distance of P from a point on C. Write two integers separated by only a comma: the integer closest to d^2, the integer closest to D^2. E.g., 20,25 is an answer in the correct format. [3 points]

Answers

(10) $2x^3 + 7x = 9$. For distance to be minimum, the line joining $(9, -3)$ with point (x, x^2) must be normal to C, so must have slope $(x^2 + 3)/(x - 9) = -1/2x$. We get $2x^3 + 7x = 9$. Alternatively, minimize $\sqrt{(x-9)^2 + (x^2+3)^2}$ over $[0, 2]$, which is easier done by throwing away the square root. One gets the same equation for a critical point, but then more is needed to argue that the *minimum* satisfies this equation. See below.

(11) $d^2 = 80, D^2 = 98$. Squared distance of a point (x, x^2) on C from P is $(x - 9)^2 + (x^2 + 3)^2$. An extremum of a differentiable function of x over a closed interval occurs at either a critical point or at an endpoint. Critical points are solutions of $2x^3 + 7x = 9$. Clearly $x = 1$ is a solution and one can visually deduce (how?) that this is the only solution and that it gives the minimum* (so for the maximum we just need to check the values at the two endpoints). Or algebraically, $2x^3 + 7x - 9 = (x - 1)(2x^2 + 2x + 9)$ and $2x^2 + 2x + 9$ has no real root. At $x = 0, 1, 2$ the values of the objective function are, respectively, $90, 80, 98$.

*Note: As $x = 1$ gives the minimum, anything of the form $ax^3 + bx = c$ with $a + b = c$ and $gcd(a, b, c) = 1$ is a valid (if cheeky) answer to (10). If you did this consciously, write to us.

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(12) Count the number of *ordered* tuples of *integers* (a, n_1, n_2, n_3, n_4) such that all three conditions below are satisfied. [4 points]

- $a > 0$.
- $n_i \geq -1$ for each i.
- $a^2 + n_1 + n_2 + n_3 + n_4 = 5$.

Answer: $222 = 165 + 56 + 1$

There are three cases: $a = 1, 2, 3$. They yield, respectively, $165, 56$ and 1 possibilities. In each case add 1 to each n_i and use the "stars and bars" method. For $a = 1$ we have $n_1 + n_2 + n_3 + n_4 = 4$, so the sum of the four non-negative integers $n_i + 1$ is 8, which can be achieved in $\binom{8+3}{3} = 165$ ways. For $a = 2$ we have $n_1 + n_2 + n_3 + n_4 = 1$, so the sum of the four non-negative integers $n_i + 1$ is 5, which can be achieved in $\binom{5+3}{3} = 56$ ways. Finally, for $a = 3$, we have $n_1 + n_2 + n_3 + n_4 = -4$, forcing each $n_i = -1$.

Information for question (13)

Claim: Suppose a, b are distinct roots of a polynomial $p(x)$. Then $p(x)$ is a multiple of $(x-a)(x-b)$.

Proof from first principles: Use long division to get $p(x) = (x-a)q(x) + r$ where __1__ is some constant. Substituting $x =$ __2__, we get $r =$ __3__, so __4__ is a multiple of $(x-a)$. Now substituting $x =$ __5__ we get that __6__ is a root of __7__. Finally, again apply the logic in the first two sentences, now to $q(x)$. This shows that __8__ is a multiple of __9__, completing the proof.

Options for the blanks

A. a	B. b	C. r
D. 0	E. $q(a)$	F. $q(b)$
G. $q(r)$	H. $p(x)$	I. $q(x)$
J. $(x-a)$	K. $(x-b)$	L. $(x-r)$

(13) Complete the given proof by writing a sequence of nine letters indicating the correct options to fill in the numbered blanks 1 to 9. Do not use any spaces, full stop or any other punctuation. E.g., **ABACDIJKB** is in the correct format. [3 points]

Answer

Proof from first principles: By using long division, $p(x) = (x-a)q(x) + r$ for some constant r. Substituting $x = a$, we get $r = 0$, so $p(x)$ is a multiple of $(x-a)$. Now substituting $x = b$ we get that b is a root of $q(x)$. Finally, again apply the logic in the first two sentences, now to $q(x)$. This shows that $q(x)$ is a multiple of $x-b$, completing the proof.

Information for question (14)

Logic similar to Question (13) works for a polynomial $P(x, y)$ in two variables: if after substituting $x = f(y)$ one gets $P(f(y), y) = 0$, then $P(x, y) = (x - f(y))Q(x, y)$. Similarly if $P(g(y), y) = 0$ as well, then an analogous conclusion holds. Assume this.

Now let $A(x, y)$ be a *symmetric* polynomial, meaning $A(x, y) = A(y, x)$. Let $\omega = e^{2\pi i/3}$. Suppose that substituting $x = \omega y$ gives $A(\omega y, y) = 0$. Then $A(x, y)$ is a multiple of __1__. Therefore it is a multiple of __2__ too. Therefore $A(x, y)$ must be a multiple of the following polynomial with integer coefficients: __3__.

Options for the blanks

A. $(x - \omega y)$
B. $(y - \omega x)$
C. $(x + \omega y)$
D. $(y + \omega x)$
E. $(x + y)$
F. $x^2 - 2xy + y^2$
G. $x^2 - xy + y^2$
H. $x^2 + y^2$
I. $x^2 + xy + y^2$
J. $x^2 + 2xy + y^2$

(14) Complete the reasoning about $A(x, y)$ by writing a sequence of three letters indicating the correct options to fill in the numbered blanks 1 to 3. E.g., **ABC** is in the correct format. [2 points]

Answer

Let $A(x, y)$ be a *symmetric* polynomial, meaning $A(x, y) = A(y, x)$. Let $\omega = e^{2\pi i/3}$. Suppose that substituting $x = \omega y$ gives $A(\omega y, y) = 0$. Then $A(x, y)$ is a multiple of $(x - \omega y)$. Therefore (by symmetry) it is a multiple of $(y - \omega x)$ too. Therefore $A(x, y)$ must be a multiple of the following polynomial with integer coefficients: $x^2 + xy + y^2 = (x - \omega y)(y - \omega x)/(-\omega)$.

Information for question (15)

Claim: $x = y = z = 0$ is the only integer solution to $x^2 + y^2 = 105z^2$.

Proof: Consider an integer solution (x, y, z) with the __1__ possible value of $|x| + |y| + |z|$.

When any perfect square is divided by 3 the possible remainders are 0 and 1. Therefore, when each term on the left hand side $x^2 + y^2$ is divided by 3, the possible combinations of remainders are $0 + 0, 0 + 1, 1 + 0$, and $1 + 1$.

It follows that __2__ must be divisible by __3__. Therefore $x^2 + y^2$ is divisible by __4__. Therefore __5__ is divisible by __6__ because 105 is divisible by __7__ but not by __8__. Now observe that __9__ x, y and z by __10__ still gives a solution of the given equation. Unless $x = y = z = 0$, this contradicts the first sentence of the proof.

Options for the blanks

A. largest	B. smallest
C. x	D. y
E. z	F. x or y
G. both x and y	H. x or y or z
I. all of x, y, z	J. dividing
K. multiplying	An integer from 0 to 9

(15) Fill each blank with the letter of an option OR an integer between 0 to 9. For example **K9AC6JK1BB** is an answer in the correct format. [4 points]

Answer

Consider an integer solution (x, y, z) with the smallest possible value of $|x| + |y| + |z|$. When any perfect square is divided by 3 the possible remainders are 0 and 1. Therefore, when each term on the left hand side is divided by 3, the possible combinations of remainders are $0 + 0$, $0 + 1$, $1 + 0$, and $1 + 1$.

It follows that both x and y must be divisible by 3. Therefore the left hand side is divisible by 9. Therefore z is divisible by 3 because 105 is divisible by 3 but not by 9. Now observe that dividing x, y and z by 3 still gives a solution of the given equation. Unless $x = y = z = 0$, this contradicts the first sentence of the proof.

Information for question (16)

Can the argument in Question (15) work with the exact same logic to prove the same claim for the same integer equation $x^2 + y^2 = 105z^2$, if we divide by a number other than 3? You are asked to test this for 5 and 7 and answer as explained in the question.

(16) Write your answer as a single letter from options A,B,C,D followed by listing in increasing order the letter of every equality that is NOT encountered while deciding the correct option from A to D. For example, **AGI** is an answer in the correct format. [2 points]

Options for (16)

A. Same argument works for 5 as well as 7
B. Same argument works for 5 but not 7
C. Same argument works for 7 but not 5
D. Same argument works for neither 5 nor 7

E. $0 + 1 = 1$ F. $1 + 1 = 2$
G. $1 + 2 = 3$ H. $1 + 4 = 5$
I. $2 + 2 = 4$ J. $3 + 3 = 6$
K. $4 + 4 = 8$

Answer

CJ. It works for 7 but not 5. Only $3 + 3 = 6$ is not enountered as no square leaves remainder 3 when divided by 5 or by 7. All others equalities are encountered as $0, 1, 4, 2$ are remainders of squares modulo 7. Note that $1 + 4 = 5$ causes the method to fail for 5. (Reason: because then LHS is divisible by 5 without x and y being divisible by 5, so the proof cannot proceed. The proof works when dividing by 7 because the only way to get a multiple of 7 by adding two numbers from the list $0, 1, 4, 9 = 2 \bmod 7$ is $0 + 0$, allowing us to proceed exactly as was done using division by 3 in the previous question.)

Information for questions (17) and (18)

Suppose a function $f(x)$ has domain $\mathbb{R}$ and satisfies the following three conditions.

(A) f is differentiable. (B) f is increasing. (C) $0 < f(x) \leq 1$ for each x.

Suppose $\lim_{x\to\infty} f'(x) =$ a real number L. Complete the following proof showing $L = 0$.

Proof: By the given condition $\underline{\quad 1 \quad}$, the value of L must be $\underline{\quad 2 \qquad 3 \quad}$. If L is nonzero, we can choose a large enough N such that for $x\underline{\quad 4 \quad}N$, the value of $f'(x)$ is $\geq \underline{\quad 5 \quad}$. Therefore by the following theorem $\underline{\quad 6 \quad}$, for any $x \geq \underline{\quad 7 \quad}$, given condition $\underline{\quad 8 \quad}$ will be violated, giving a contradiction.

Options for the blanks

A. (A)	B. (B)	C. (C)
D. $=$	E. $\geq$	F. $\leq$
G. 0	H. 1/2	I. 1
J. $L/2$	K. $2L$	L. L
M. $N + \frac{L}{2}$	N. $N + L$	O. $N + 2L$
P. $N + \frac{2}{L}$	Q. $N + \frac{1}{L}$	R. $N + \frac{1}{2L}$

S. Mean value theorem

T. Extreme value theorem

U. Intermediate value theorem

V. Fundamental theorem of calculus

Questions

(17) Write a sequence of 5 letters indicating the correct options to fill in the numbered blanks 1 to 5. For example, **BACDE** is in the correct format. [2 points]

(18) Write a sequence of 3 letters indicating the correct options to fill in the numbered blanks 6 to 8. E.g., **WJE** is in the correct format. [2 points]

Answers

By the given condition (B), the value of L must be ≥ 0. If L is nonzero, we can choose a large enough N such that for each $x \geq N$, the value of $f'(x)$ is $\geq L/2$. Therefore by the mean value theorem, for $x \geq N + 2/L$, condition (C) will be violated, giving a contradiction. Reasoning: (1) For $x \geq (N+2/L)$, by MVT, $f(x) - f(N) = f'(\text{a number} > N)(x - N) \geq (L/2)(2/L) = 1$, forcing $f(x) > 1$ as $f(N) > 0$. OR (2) If $f'(x)$ is known to be Riemann integrable (which is true, e.g., if f' is known to be continuous), then instead of MVT one can also argue using the fundamental theorem of calculus as follows. $f(x) - f(N) = \int_N^x f'(x)\,dx \geq (L/2)(2/L)$, since the integrand is at least $L/2$ over an interval of length at least $2/L$.

Note: this question is a small prelude to problem B5. See the answer to B5 for more.

Draft answer key to part A

(2) 2

(3) 40

(4) 60

(5) 79

(6) 9

(7) 1.41

(8) 2

(9) 50

(10) 2,7,9

(11) 80,98

(12) 222

(13) CADHBBIIK

(14) ABI

(15) BG39E339J3

(16) CJ

(17) BEGEJ

(18) SPC (VPC also accepted)

B1. [12 points] Suppose five complex numbers z_1, z_2, z_3, z_4, z_5 form the vertices of a regular pentagon that is inscribed in a circle of radius 2 with center at $c = 6 + 8i$.

(a) Find all possible values of $S = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2$. State a value of z_1 maximizing $|S|$.

(b) Find all possible values of $P = z_1 z_2 z_3 z_4 z_5$. State a value of z_1 minimizing $|P|$.

Solution

All five $z_j - c$ are on the circle with radius 2 and center at the origin. Let $z_1 - c = 2e^{i\theta}$. So we have $\{z_j - c \,|\, j = 1, 2, 3, 4, 5\} = \{2e^{i(\theta+(2\pi j/5))} \,|\, j = 1, 2, 3, 4, 5\}$. Thus the five z_j are the five complex roots of the degree 5 polynomial $(z - c)^5 - 32e^{5i\theta}$. So

$$(z - c)^5 - 32e^{5i\theta} = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5).$$

Use Vieta's formulas (i.e., expand both sides and compare coefficients of powers of z) to get

$$z_1 + z_2 + z_3 + z_4 + z_5 = 5c \qquad \sum_{1\le j<k\le 5} z_j z_k = 10c^2 \qquad \text{and} \qquad z_1 z_2 z_3 z_4 z_5 = c^5 + 32e^{5i\theta}.$$

(a) $S = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = (\sum_{1\le j\le 5} z_j)^2 - 2\sum_{1\le j<k\le 5} z_j z_k = 25c^2 - 20c^2 = 5c^2$ is independent of θ, i.e., constant regardless of the placement of the inscribed pentagon. So any z_1 on the given circle will work, e.g., $z_1 = 8 + 8i$ or $6 + 10i$, etc. $|S| = 5|c|^2 = 500$.

(b) We have $P = c^5 + 32e^{5i\theta}$. As θ varies, the resulting set of points forms a circle of radius 32 with center at $c^5 = (6+8i)^5$. The minimum value of $|P|$ is achieved when P is the intersection point of this circle with the segment joining the origin with c^5, giving $|P| = |c^5| - 32 = 99968$.

To describe a value for z_1 (i.e., a value for θ) minimizing $|P|$, let $c = 10e^{i\alpha}$, so $c^5 = 10^5 e^{5i\alpha}$. Then maximum/minimum value of $|P|$ is attained precisely when c^5 and $32e^{5i\theta}$ are, respectively, along the same/opposite rays from the origin. This in turn is equivalent to $5(\alpha - \theta)$ being an even/odd multiple of π. For example, taking $\theta = \alpha + (\pi/5)$, i.e., $z_1 = c + 2e^{i(\alpha+(\pi/5))}$ will give $|P| = 99968$.

This can also be seen algebraically. We have $\big||a| - |b|\big| \le |a + b| \le |a| + |b|$ by the triangle inequality for complex numbers a, b. Let $b \ne 0$. The upper/lower bound for $|a+b|$ is achieved when a/b is a real number. So we have $|c^5| - |32e^{5i\theta}| \le |c^5 + 32e^{5i\theta}| \le |c^5| + |32e^{5i\theta}|$, giving $99968 \le |P| \le 100032$. The upper/lower bound is achieved when $c^5/(32e^{5i\theta}) = 5^5 e^{5i(\alpha-\theta)}$ is real, which happens precisely when $5(\alpha - \theta)$ is a multiple of π.

B2. [**12 points**] Consider the following functions from $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$ to itself. Let R_α be counterclockwise rotation by angle α and let F_α = reflection in the line that makes counterclockwise angle α with the X-axis. E.g., $R_{90^\circ}(1,0) = (0,1)$ and $F_{90^\circ}(1,0) = (-1,0)$.

(a) Evaluate $R_\alpha(r\cos\theta, r\sin\theta)$ and $F_\alpha(r\cos\theta, r\sin\theta)$, where $r \geq 0$ and θ is any angle.

(b) Geometrically describe the composition of functions $F_\alpha \circ F_0$. (Note that F_0 = reflection in the X-axis.) You may use any valid method as long as you explain clearly.

(c) For any angles α and β, geometrically describe $F_\alpha \circ F_\beta$ in one crisp sentence. You may appeal to (b) if you justify why and how your work in (b) can be used here. Find $G(P)$ for $G = (F_{20^\circ} \circ F_{25^\circ})^9 = (F_{20^\circ} \circ F_{25^\circ})$ composed with itself 9 times and P = the point $(20,25)$.

Solution

Let the origin be O and the point $Q = (r\cos\theta, r\sin\theta)$. Both R_α and F_α preserve the distance r from the origin, so one just has to find out the angle from the positive X-axis of the new ray after rotating/reflecting the ray OQ.

(a) The ray OQ is obtained by rotating the positive X-axis counterclockwise by angle θ. The rotation R_α rotates ray OQ further by angle α, taking it to the ray making counterclockwise angle $\alpha + \theta$ from the positive X-axis. So $R_\alpha(r\cos\theta, r\sin\theta) = (r\cos(\alpha+\theta), r\sin(\alpha+\theta))$.

For the reflection F_α, *draw a picture* to see the following. Reflection of $Q = (r\cos\theta, r\sin\theta)$ in the described line sends ray OQ to a ray OQ' that makes counterclockwise angle $2\alpha - \theta$ from the positive X-axis. Reason: Ray OQ makes counterclockwise angle $\theta - \alpha$ from (one half of) the reflecting line. So after reflection, the *clockwise* angle from the (same half of the) reflecting line to OQ' must also be $\theta - \alpha$. Altogether the counterclockwise angle of ray OQ' from the positive X-axis is $\alpha - (\theta - \alpha) = 2\alpha - \theta$. (Convince yourself that the reasoning is valid for all angles α, θ regardless of their signs and magnitudes.) Therefore

$$F_\alpha(r\cos\theta, r\sin\theta) = (r\cos(2\alpha-\theta), r\sin(2\alpha-\theta)).$$

(b) By (a), $F_\alpha \circ F_0(r\cos\theta, r\sin\theta) = (r\cos(2\alpha+\theta), r\sin(2\alpha+\theta)) = R_{2\alpha}(r\cos\theta, r\sin\theta)$, so $F_\alpha \circ F_0 = R_{2\alpha}$. *It is fun and worthwhile to do this by pure geometry without using coordinates.*

(c) $F_\alpha \circ F_\beta = R_{2(\alpha-\beta)}$ again by (a) or by the exact same geometric argument you were exhorted to do in (b): the composition of two reflections is a rotation by an angle determined solely by the angle between the two lines of reflection. Therefore $(F_{20^\circ} \circ F_{25^\circ}) = R_{-10^\circ}$, so $G = R_{-90^\circ}$ and $G(20,25) = (25,-20)$.

Remarks: The calculations in this problem are fundamental. Such considerations underlie mathematical study of symmetry. Here are some further easy questions in a similar vein.

(d) How do $F_\alpha \circ F_\beta$ and $F_\beta \circ F_\alpha$ compare? Observe that they are inverse functions of each other. Does F_α have an inverse? If so, what is it? With this information, again relate $F_\alpha \circ F_\beta$ and $F_\beta \circ F_\alpha$. Can you quickly calculate the composition $F_\alpha \circ R_\gamma$ where α, γ are arbitrary angles? Try to do something to the equation from (c): $F_\alpha \circ F_\beta = R_\gamma$ where $\gamma = 2(\alpha - \beta)$.

(e) Start with $\{F_0, F_{60^\circ}\}$. Compose functions from this set with each other. Add the resulting functions to the set. Keep repeating. Show that this process stops with a finite set S after which you get no new functions. What is $|S|$? Write down an $|S| \times |S|$ "composition table". Now replace F_{60° by some F_α and play the same game. Do you always end with a finite set?

This is a beginning towards group theory. See Hermann Weyl's book *Symmetry* for more.

B3. [**12 points**] A particle starts at (0,0) and travels in the first quadrant along a straight line. It maintains the slope AND stays in the square bounded by the lines $x = 0, x = 1, y = 0$ and $y = 1$ as follows. Whenever it reaches the boundary of this square it magically jumps 1 unit to the left or down or both as applicable. In other words, for $a < 1$, it jumps to $(a, 0)$ upon reaching $(a, 1)$, it jumps to $(0, a)$ upon reaching $(1, a)$, and it jumps to (0,0) if it reaches (1,1). If the particle ever reaches a previously visited point, it stops.

For example, $(0, 0) \to (1/2, 1) \xrightarrow{jump} (1/2, 0) \to (1, 1) \xrightarrow{jump} (0, 0)$ is a trajectory of length $\sqrt{5}$. (Jumps don't count for length.) Another way to visualize this trajectory is to let the particle continue across the boundary $y = 1$ and interpret what happens.

(a) What are all the possible stopping points for finite trajectories?

(b) If the particle starts at the angle 30° with respect to the X-axis, show that it never stops.

(c) Find the two smallest possible *integer* lengths among all finite trajectories.

Solution

(a) Only the starting point (i.e., the origin) can be the stopping point. Because the slope stays constant, once the slope is known, any point in a trajectory has a uniquely determined past (determined solely by its position). So if any position other than the starting point were to repeat, there would be previous points that also repeated. (This is true not just for interior points, but those on the boundary too, because the magic jump rule allows one to trace back.) But then we have a contradiction to the stopping rule.

(b) After starting, the trajectory first meets the boundary at $(1, 1/\sqrt{3})$ and then the particle jumps to $(0, 1/\sqrt{3})$. Any time the trajectory meets the boundary, the coordinate other than 1 or 0 of the meeting point will be always be irrational because the slope is irrational. So the particle can never return to the origin.

(c) Use the other suggested way to visualize a trajectory: let the particle continue in the starting straight line without being confined to the square. The prescribed path is easily deduced from the resulting magicless ray in the plane, say L. *Crucial observation:* the particle repeats a position if and only if L meets an integer point, which it will do precisely when the slope is rational, and in that case the first nonzero integer point (a, b) the particle reaches must satisfy $gcd(a, b) = 1$. The answer is 5 and 13 because these are the smallest hypotenuse lengths of integer right triangles with coprime sides, namely triangles with sides 3-4-5 and 5-12-13. (Note: 1 is not an answer because under the rule as worded, the particle cannot travel along either axis. It cannot step off the origin in those directions.)

Remarks: Suppose an ant starts walking in some direction at a point on the surface of a *donut* and continues in the same direction. Will it return to a previously visited point? The exam problem formulates and analyzes this question in terms of plane geometry. There are two steps. (1) The surface of a donut can be thought of as obtained by taking a (stretchable) square sheet of paper, first rolling it to make a cylinder by gluing two parallel edges, and then gluing together the two end circles of this cylinder to make "hollow donut". This is what the "magic jumps" in the problem are doing. Note that at the end, the four corners of the square are glued together, every other point of the border is glued to exactly one other point on the parallel edge, and no point in the interior is glued to any other point. (2) Now go a step further by starting with the entire XY plane with a grid of lines dividing the plane into squares. Define a function from the plane onto the donut so that each point on the donut is represented by infinitely many points in the plane. All points on the grid lines are appropriately mapped to the edges of the single square in the first step, which are themselves glued with the opposite edges as explained in the first step. In particular all integer points are mapped to the origin. This second step is very useful to visualize the ant's path on the torus by means of an ordinary straight line in the plane. Having multiple points in the plane represent the same point on the donut helped in translating a problem about the donut to a more tractable one about the plane.

The strategy of allowing the same object to be represented by different representatives is very useful and employed frequently in mathematics. You have already encountered this in arithmetic, e.g., when we say that the pairs (1,2) and (5,10) represent the same rational number and in modular arithmetic when we say that 7 and 37 are the same modulo 10. This problem is just one example of how the idea can be useful in geometry as well.

B4. [**12 points**] The domain of f is the set of *positive integers* and $f(xy) = f(x) + f(y)$ for all x, y. Answer the *independent* questions below. (Data from (a) are not valid for the rest.)

(a) Suppose $f(2025) = 0, f(20) = 10$ and $f(25) = 20$. What is the smallest n for which $f(n)$ is not uniquely determined? Write values of $f(x)$ for each positive integer $x < n$.

(b) Is there such a function f for which $f(x) = 0$ for all positive integers $x < 2025^{2025}$ but f is not identically 0? Show how to define such f or show that it is not possible.

(c) The domain of a function g is the set of *positive rational* numbers, *codomain the set of integers* and $g(xy) = g(x) + g(y)$ for all positive rational x, y. Suppose $g(a) = 24$, $g(b) = 2025, g(c) = 10^{2025}$ for some rational numbers a, b, c. Show that there are infinitely many rational numbers r such that $g(r) = 1$. (Note: "*codomain the set of integers*" was missing in the exam, so part (c) will be graded by taking this into account. Details below.)

Solution

Such f is uniquely defined by assigning for each prime p an arbitrary value in the codomain, say λ_p, to $f(p)$. Then $f(\prod_p p^{k_p}) = \sum_p k_p\lambda_p$ and unique factorization of positive integers shows that the function is well defined. The required property is easily checked for all positive integers x, y.

(a) $f(1) = f(1) + f(1)$ so $f(1) = 0$. Now $f(25) = 2f(5) = 20$ gives $f(5) = 10$. Then $f(20) = f(2^2 \times 5) = 10$ forces $f(2) = 0 = f(4)$ and $f(2025) = f(3^4 \times 25) = 0$ forces $f(3) = -5$. Then $f(6) = f(3) + f(2) = -5$ as well. The first undetermined value is $f(7)$, which can be arbitrary by the previous paragraph.

(b) Yes. Define $f(p) = 0$ for all primes $p < 2025^{2025}$ and $f(p) \neq 0$ for at least one prime $p > 2025^{2025}$, which is possible to do because there are infinitely many primes.

(c) $g(a^x b^y c^z) = 24x + 2025y + 10^{2025}z$ for any *integers* x, y, z (including negative ones). This is easy to see using $g(1/a) + g(a) = g((1/a)a) = g(1) = 0$. Now $gcd(24, 2025, 10^{2025}) = 1$ by looking at prime factorizations $24 = 2^3 \times 3, 2025 = 3^4 \times 5^2$ and $10^{2025} = 2^{2025} \times 5^{2025}$. So by the Bézout descripton of *gcd* as a linear combination, there is an integer combination $24x + 2025y + 10^{2025}z = 1$. For such integers x, y, z, take $r = a^x b^y c^z$ to get $g(r) = 1$.

The following portion of (c) about showing infinitely many r is taken off the exam. Only existence of one r shown in the previous paragraph is enough for full credit. To find infinitely many r with $g(r) = 1$ is equivalent to finding a positive rational $s \neq 1$ such that $g(s) = 0$. Reason: $g(s) = 0$ gives $g(rs^k) = 1$ and rs^k are all distinct as k ranges over $\mathbb{Z}$. Conversely $g(r_1) = g(r_2)$ gives $g(r_1/r_2) = 0$. How to get such s? If $g(p) = 0$ for some prime p, take $s = p$. Otherwise take primes $p \neq q$. Then $s = p^{g(q)}/q^{g(p)}$ works. This s is rational because $g(p)$ and $g(q)$ are *integers* and it is not 1 unless $g(p), g(q)$ are both 0.

Remark: In fact without some restriction on the codomain, $s \neq 1$ with $g(s) = 0$ need not exist, so the "infinitely many" part of the question is erroneous without additional hypothesis. Here is a counterexample by taking the codomain to be $\mathbb{R}$ (consistent with the overall convention in the exam). We want a function g such that $g(s) = 0$ only for $s = 1$ while satisfying other given conditions in (c). For $s = \prod_p p^{k_p}$ with k_p integers, $g(s) = g(\prod_p p^{k_p}) = \sum_p k_p \lambda_p$. The first paragraph of the solution stays valid for g because unique factorization is valid for positive rational numbers too (by allowing negative powers of primes). So we need to find a family of numbers λ_p in the codomain such that $\sum_p k_p \lambda_p = 0$ *only* when *all* integers k_p are 0. This amounts to saying that all λ_p are linearly independent when $\mathbb{R}$ is regarded as a vector space over $\mathbb{Q}$. By linear algebra, $\mathbb{R}$ has a $\mathbb{Q}$-basis, which must be uncountable. (Reason: The $\mathbb{Q}$-span of a finite/countable set of real numbers is countable, while $\mathbb{R}$ is not.) So the numbers λ_p can be taken to be part of any such basis. As a concrete example let $\lambda_2 = 1$ (with $a = 2^{24}, b = 2^{2025}, c = 2^{10^{2025}}$ so as to satisfy the part (c) hypothesis), and let $\lambda_p = \sqrt{p}$ for odd primes p. It needs proof that this works but it does.

B5. [16 points] Solve the following. Part (b) can be done independently and may be easier.

(a) Construct a function f with domain $\mathbb{R}$ such that f is differentiable, weakly increasing, bounded, and $\lim_{x\to\infty} f'(x)$ does not exist. (*Weakly increasing* means $f(a) \leq f(b)$ for $a < b$. *Bounded* means there are constants m, M such that for every real x, we have $m < f(x) < M$.)

Possible hints: What kind of function should $f'(x)$ be to satisfy the requirements? Thinking in terms of pictures may help. Is there a way to construct a function whose derivative is a desired function? If needed, you may take the domain of f to be $[0, \infty)$ instead.

(b) Construct a function g with domain $\mathbb{R}$ such that g is differentiable, *strictly* increasing, bounded, $\lim_{x\to-\infty} g(x) = 0$, $\lim_{x\to\infty} g'(x)$ does not exist, and $\lim_{x\to-\infty} g'(x)$ does not exist. (*Strictly increasing* means $g(a) < g(b)$ for $a < b$.)

Possible hint: You may use your answer to part (a) and adjust as necessary. Even if you did not do part (a), you may take as given a function f with domain $[0, \infty)$ and the required properties in (a). Then show with clear explanation how to build g in terms of f.

Solution

(a) To meet all requirements it is enough to arrange all of the following.

1. $f'(x)$ is always ≥ 0. This ensures that $f(x)$ is weakly increasing everywhere.
2. $f'(x)$ is continuous, so the fundamental theorem of calculus applies and for any x we have $f(x) = f(0) + \int_0^x f'(t)\,dt$.

3. $\int_{-\infty}^{\infty} f'(t)\,dt = A$ is finite. This ensures that f is bounded. Proof: $A = B + C$, where $B = \int_{-\infty}^{0} f'(t)\,dt$ and $C = \int_{0}^{\infty} f'(t)\,dt$. Because the integrand $f'(x) \geq 0$ everywhere, we have the following bounds. For $r > 0$, $f(r) = f(0) + \int_0^r f'(t)\,dt \leq f(0) + C$. For $s < 0$, $\int_s^0 f'(t)\,dt = f(0) - f(s) \leq B$, giving $f(0) - B \leq f(s)$. Combining with $f(s) \leq f(r)$ (because f is increasing) we deduce that the range of f is contained in the window $[f(0) - B, f(0) + C]$ of width A. (We're just using 0 as a helper point.)

4. $f'(x)$ does not have a limit at ∞. One way to arrange this is to pick two distinct nonnegative numbers, say 0 and 1, and ensure that for any N, $f'(x)$ attains each of the two values for some $x > N$.

Here is an example putting all of this together. (Other schemes are possible.) Define a continuous nonnegative function h as follows. For each nonzero integer n, on the segment $[n - 1/2^{|n|}, n + 1/2^{|n|}]$ on the X-axis, define $h(x)$ so that its graph is two sides of the triangle with vertices at the points $(n \pm (1/2^{|n|}), 0)$ and $(n, 1)$. Note that the area of this triangle is $1/2^{|n|}$. For all $(x, 0)$ not on the base of any of these triangles, define $h(x) = 0$. Then define $f(x) = \int_0^x h(t)\,dt$. See that $f'(x) = h(x)$ meets conditions 1 to 4. Here $B = C = 1 =$ total area of triangles on either side of the Y-axis. (Note that $\lim_{x \to -\infty} h(x)$ also fails to exist. This was done anticipating the added requirement in part (b). For part (a) alone we could just as well have put the triangles only on the positive side and defined $h(x) = 0$ for $x \leq \frac{1}{2}$.)

(b) Take f constructed in part (a) and add to it a *strictly* increasing bounded differentiable function $k(x)$ such that $k'(x)$ does have a limit at ∞ and at $-\infty$, e.g., $k(x) = \arctan(x)$, or $k(x) = 1 - e^{-x}$ for positive x and $e^x - 1$ (reflection in the origin) for negative x. Finally, arrange the limit at $-\infty$ to be 0 via a vertical shift by subtracting $L = \lim_{x \to -\infty}(f(x) + k(x))$. (Why does this limit exist?) Altogether $g(x) = f(x) + k(x) - L$.

If one uses (a) as a black box, then more work is required. Take a function f with domain $[0, \infty)$ meeting the requirements in part (a). (If f it is defined for negative x, just discard that part.) Now shift vertically to ensure $f(0) = 0$. Then define the function for $x < 0$ so as to make it odd: $f(x) = -f(-x)$. Note that f is has right hand derivative at 0 by the black box, so the new function is still differentiable at 0 and hence everywhere. Reflection has ensured that $\lim_{x \to -\infty} f'(x)$ also fails to exist. Now continue as in the previous paragaph.

Remark: This problem is based on the following question raised by a student at CMI. *For an increasing, differentiable and bounded function $f : \mathbb{R} \to \mathbb{R}$, must it be true that $\lim_{x \to \infty} f'(x) = 0$?* As seen in questions 17-18 in part A, if this limit exists, it must be 0, bringing us to the situation in this problem.

B6. [16 points] $a_1, a_2, a_3, \ldots, a_n$ is a sequence of *distinct* numbers, each written on a card. Take the cards one by one in order. Place them into stacks subject to this rule: *one can only place a smaller number on top of a larger number or one can start a new stack with a card.*

Here is a simple greedy strategy. Make stacks along a line and *place the incoming card on top of the leftmost stack possible.* So if the incoming card cannot be placed on any existing stack, then we use that card to start a new stack to the right of all current stacks.

For example, with the sequence 3 7 2 5 6 4 9 8 we get the four stacks:

	4		
2	5		8
3	7	6	9

(a) Show that under the greedy strategy, the top numbers on the stacks increase from left to right, e.g., 2 4 6 8 in the example above.

(b) Show that under the greedy strategy, the number of stacks is the length of a longest possible increasing subsequence of $a_1, a_2, a_3, \ldots, a_n$. A subsequence means $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. The example given above has an increasing subsequence of length 4 (e.g., 3 5 6 8 among others) but none of length 5. It also gives 4 stacks as claimed.

Possible hint: for an entry x in the given sequence, let $\ell(x) =$ the length of a longest increasing subsequence whose last entry is x. What are the values of $\ell(x)$ in the example?

(c) Show that the greedy strategy gives the minimum possible number of stacks.

(d) Find two sequences that give the same end result of stacks after using the greedy strategy.

(e) Show that not every sequence of legal stacks is obtainable by using the greedy strategy. (*Legal stack* means numbers increase from top to bottom.) Given a sequence of legal stacks along a line, how will you decide if it arises as the result of using the greedy strategy on some sequence of numbers?

Solution

Use induction to prove (a) to (c). Note that by the very nature of these statements, in order to be true, they have to be true at every stage of the described procedure.

(a) An incoming number x will skip stacks whose top numbers are less than x and go on top of the first y (if any) bigger than x. Top entries of stacks to the right of y (if any) are bigger than y by induction on the length of the sequence, so bigger than x.

(b) Claim: $\ell(x) =$ the number, say $n(x)$, of the stack in which x is placed.

Proof: We prove $\ell(x) \geq n(x)$ by induction on the length of the sequence. If $n(x) = 1$ then x is smaller than all previous entries by (a). Otherwise let $y =$ the top entry in stack $n(x) - 1$ at this point. We know $y < x$ by (a). By induction there is an increasing subsequence of length $n(x) - 1$ ending in y. Append x to this.

For $n(x) \geq \ell(x)$, do induction on $\ell(x)$. (Or use the more basic argument in (c) below.) Suppose $\ell(x) = k$ with increasing subsequence $c_1 < \ldots < c_k = x$. If $k = 1$, then x is smaller than all previous entries so must enter the first stack. Else by induction c_{k-1} must have entered stack $k - 1$ or greater. Every subsequent change in any stack will only make the top entry smaller. So $x = c_k$, being bigger than c_{k-1}, must enter a stack to the right of c_{k-1}.

(c) If $c_1 < c_2 < \ldots < c_k$ is an increasing subsequence then all c_i must be in different stacks no matter how the game is played. Prove this by induction on k. When c_k comes the previous $k-1$ entries c_i are in $k-1$ different stacks (by induction), so the top numbers of those stacks are all less than c_k, which must therefore go to a different stack.

Therefore the number of stacks must be at least as much as the length of a longest increasing subsequence. But by (b) the greedy strategy produces exactly these many stacks.

(d) 2 1 3 and 2 3 1 both yield the same result. (Generally if three consecutive entries yxz in a sequence satisfy $x < y < z$ then see that switching yxz to yzx leaves the result unaltered.)

(e) By part (a), for an attainable configuration the top entries of the stacks have to be increasing from left to right. E.g., 2 followed by stack 1 3 cannot be attained. Even simpler, a sequence of stacks of height 1 is attainable only if the numbers increase from left to right.

The condition in part (a) is also sufficient by induction on the length. Construct the sequence backwards: remove the top entry of the rightmost stack and use it as the last entry in the sequence. The stated condition stays valid for the remaining configuration. Iterate.

Remark: This problem is based on patience sorting. It is also related to the Robinson-Schensted-Knuth correspondence, a beautiful and important chapter in combinatorics.