NCM IST, Mathematics for Computer Science
Convex functions, multiplicative weight updates

26 June, 2018

1. For two vectors a, b in $\mathcal{R}^n$ show that $2 < a, b >= ||a||^2 + ||b||^2 - ||a-b||^2$.

2. Using the analysis done in class for a single function f, show that there is an algorithm which, given a starting point x_1, and parameters ϵ and D, and a family $f_1, \ldots, , f_T$ of convex functions which are G-Lipschitz, produces a sequence ofpoints $x_1, x_2, \ldots, x_T$ such that

$$\frac{1}{T}\sum_{t=1}^{t=T}(f_t(x_t) - f_t(x*)) \leq \frac{DG}{\sqrt{T}}$$

for any $x*$satisfying $||x_1 - x*|| \leq D$ and T being $(\frac{DG}{\epsilon})^2$. Your algorithm should produce x_{t+1} knowing only $f_1, \ldots, f_t$.

3. Show that the strategy of following the opinion of the majority among the experts on a given day has very large regret.

4. Show that $-\ln(1-x) \leq x + x^2$ for $|x| \leq 1/2$.

5. Consider the problem of learning weights which fit a data set with inputs X in $\mathcal{R}^D$ and outputs in $\mathcal{R}^k$. We wish to find W such that $Y = WX + B$ for appropriately sized matrices. Solve this analytically.

6. **Convex set** A set $K \subseteq \mathbb{R}^n$ is convex if, for every two points in K, the line segment connecting them is contained in K, i.e., $\forall x, y \in K$ and $\forall \lambda \in [0,1]$ we have $\lambda x + (1-\lambda)y \in K$.

   Convex function Def 1 A function $f : K \to \mathbb{R}$ is convex if $\forall x, y \in K$, $\forall \lambda \in [0,1]$,

   $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

   Convex function Def 2 A function $f : K \to \mathbb{R}$ is convex if $\forall x, y \in K$,

   $f(y) \geq f(x) + < \bigtriangledown f(x), y - x >$

   Show that both the definitions of convex functions are equivalent. Show that a local minima of a convex function is a global minima.

7. **Strictly convex function:** A function $K \to \mathbb{R}$ is strictly convex, if $\forall x, y \in K$ and $x \neq y$, $\forall \lambda \in (0,1)$,

   $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$

   Show that a strictly convex function has a unique minima.