Lecture 3: Shor code

· Recap the table from lecture #1.

Suppose we have both bit and phase flip errors:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{1-p} \alpha|0\rangle + \beta|1\rangle \quad \text{No error}$$

$$\xrightarrow{p/3} \alpha|1\rangle + \beta|0\rangle \quad \text{Bit flip error: } X$$

$$\xrightarrow{p/3} \alpha|0\rangle - \beta|1\rangle \quad \text{Phase flip : } Z$$

$$\xrightarrow{p/3} \alpha|1\rangle - i\beta|0\rangle \quad \text{Bit and Phase flip: } Y$$

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3} X\rho X + \frac{p}{3} Z\rho Z + \frac{p}{3} Y\rho Y \quad : \text{Depolarizing channel.}$$

Recall that we can correct Bit and Phase flip separately using QEC codes:

(i) $|0\rangle \longrightarrow |000\rangle$, $|1\rangle \longrightarrow |111\rangle$ — Corrects Bit flip

(ii) $|0\rangle \longmapsto |{+}{+}{+}\rangle$, $|1\rangle \longmapsto |{-}{-}{-}\rangle$ — Corrects phase flips

QUESTION: Can we combine these two strategies to correct X and Z errors?

$$|0\rangle \longrightarrow |{+}{+}{+}\rangle = (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)/\sqrt{8}$$

$$\underbrace{|1\rangle \longrightarrow |{-}{-}{-}\rangle = (|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)/\sqrt{8}}$$

Step 1: Protect against phase flip (Z)

$$|0\rangle \longmapsto |000\rangle$$
$$|1\rangle \longmapsto |111\rangle$$

Step 2: Protect against Bit flips

$$|0\rangle \longrightarrow |{+}{+}{+}\rangle = (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)$$

$$\underbrace{|1\rangle \longrightarrow |{-}{-}{-}\rangle = (|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)}$$

Step 1: Protect against phase flip (Z)

$$|\bar{0}\rangle = \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{\sqrt{8}}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \mapsto |\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$$

- Hence the Shor's code is given by:

$$|0\rangle \mapsto |\bar{0}\rangle = \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{\sqrt{8}}$$

$$|1\rangle \mapsto |\bar{1}\rangle = \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{\sqrt{8}}$$

We are using $n=9$ physical qubits to encode $k=1$ logical qubit.

- Span $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ defined above is the Shor's code.

Original paper: Phys. Rev. A, 52, R2493 (R). Published 1 October 1995.

Analyzing the error correcting capabilities of the Shor's code

$$|\psi\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$$

$$= \alpha \frac{(|000\rangle+|111\rangle)^{\otimes 3}}{2\sqrt{2}} + \beta \frac{(|000\rangle-|111\rangle)^{\otimes 3}}{2\sqrt{2}}$$

$m_1=+1$, $m_2=+1$, $m_3=+1$, $m_4=+1$, $m_5=+1$, $m_6=+1$

$m_7=+1$, $m_8=+1$

If there is a bit flip on qubit in block 'b'

Block b: $\frac{|000\rangle+|111\rangle}{\sqrt{2}} \xmapsto{X_1} \frac{|100\rangle+|011\rangle}{\sqrt{2}}$

If we measure Z_1Z_2 and Z_2Z_3 in block 'b':

$m_1=-1$, $m_2=+1$.

Task: Measure Z_iZ_j in every block and treat each block as a bit flip code.

Hence the operators to be measured are:

$$\underset{m_1}{Z_1Z_2},\ \underset{m_2}{Z_2Z_3},\ \underset{m_3}{Z_4Z_5},\ \underset{m_4}{Z_5Z_6},\ \underset{m_5}{Z_7Z_8},\ \underset{m_6}{Z_8Z_9}$$

If there is a phase flip error on qubit in block 'b':

$$\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)^{\otimes 3} \xmapsto{Z_1} \left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)^{\otimes 2}$$

If we measure $X_1X_2X_3X_4X_5X_6$: measuring the parity of the relative phases in two blocks.

m_7: eigenvalue of $\underbrace{X_1X_2X_3}_{\text{block 1}}\underbrace{X_4X_5X_6}_{\text{block 2}}$

m_8: eigenvalue of $\underbrace{X_4X_5X_6}_{\text{block 2}}\underbrace{X_7X_8X_9}_{\text{block 3}}$

For the Z_1 error, we find:

$m_7=-1$, $m_8=+1$.

- Summary of measurements to gather information on bit flip errors:

$$Z\otimes Z\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1 \; : \; Z_1 Z_2$$
$$1\otimes Z\otimes Z\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1 \; : \; Z_2 Z_3$$
$$1\otimes 1\otimes 1\otimes Z\otimes Z\otimes 1\otimes 1\otimes 1\otimes 1 \; : \; Z_4 Z_5$$
$$1\otimes 1\otimes 1\otimes 1\otimes Z\otimes Z\otimes 1\otimes 1\otimes 1 \; : \; Z_5 Z_6$$
$$1\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\otimes Z\otimes Z\otimes 1 \; : \; Z_7 Z_8$$
$$1\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\otimes Z\otimes Z \; : \; Z_8 Z_9$$
$$X\otimes X\otimes X\otimes X\otimes X\otimes X\otimes 1\otimes 1\otimes 1 \; : \; X_1 X_2 X_3 X_4 X_5 X_6$$
$$1\otimes 1\otimes 1\otimes X\otimes X\otimes X\otimes X\otimes X\otimes X \; : \; X_4 X_5 X_6 X_7 X_8 X_9$$

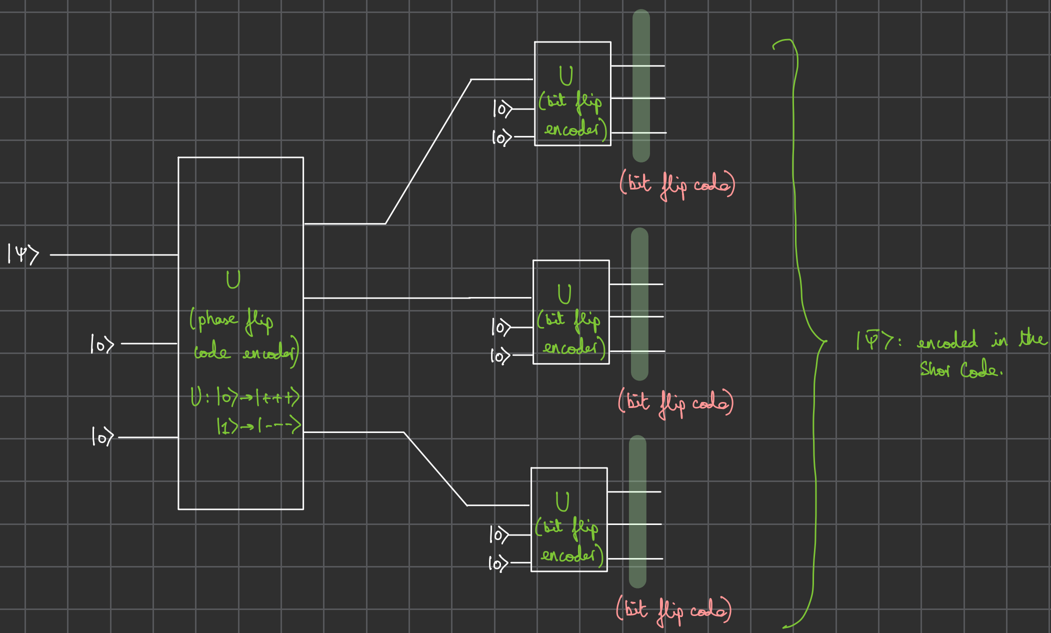
Note: Any of the above operations preserve the states $|\bar{0}\rangle$ and $|\bar{1}\rangle$. Their eigenvalues on a noisy state $E|\bar{\Psi}\rangle$ tells us information about the errors.

Quantum circuit for QEC using the Shor code:

Let us see how to prepare the encoded states using a quantum circuit: encoder U:

Note:
$$U|0\rangle \longmapsto |\bar{0}\rangle = \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}$$
$$U|1\rangle \longmapsto |\bar{1}\rangle = \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}}$$

Observation!
$|\bar{0}\rangle = \left(\text{Each qubit of } |{+}{+}{+}\rangle \text{ encoded in the bit flip code}\right)$
$|{-}\rangle = \left(\text{Each qubit of } |{-}{-}{-}\rangle \text{ encoded in the bit flip code}\right)$



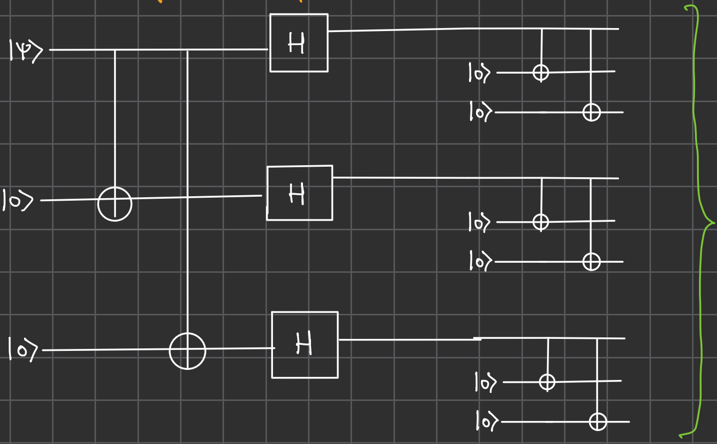
Important error correction terminology:

- The number of physical qubits : n $\equiv$ size / block length of the code
- Dimension of the encoded space : span $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ $\equiv$ k : number of logical qubits
- Quantum code : vector space spanned by $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ of dimension 2^k.
- Measuring operators to detect errors.
  - For each operator O_i, its eigenvalue measured is m_i

    State after measuring: $\dfrac{(\mathbb{1} + (-1)^{m_i} O_i)}{2} |\psi\rangle$.

  - Mapping $m_i \to s_i$ where $(-1)^{s_i} = m_i$, gives us a binary sequence $(s_1, \ldots, s_m)$: Error syndrome.

The full encoding circuit for Shor's code is:

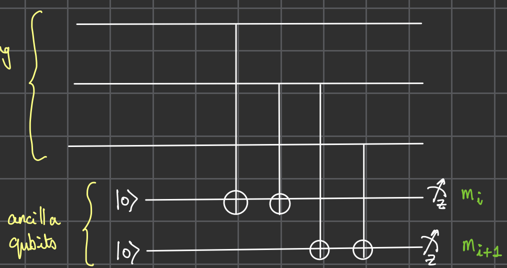


$|\bar{\psi}\rangle$: encoded in Shor's code.

Next, let us examine the circuit for doing QEC: measurments of operators to gather infor- -mation on the errors:

- we have 8 measurements to do:
  - (i) 6 measurements of $Z_i Z_{i+1}$ operators in each of the 3 codeblocks
    - similar to the bit flip code's measurements, we do

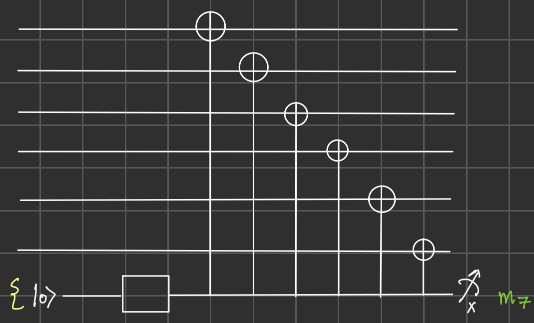
qubits belonging to block b ($b \in \{1,2,3\}$). We will measure $Z_j Z_{j+1}$, $Z_{j+1} Z_{j+2}$



  - (ii) 2 measurements, to correct phase flip errors:
    - measuring $X_1 X_2 X_3 X_4 X_5 X_6$: obtaining the outcome $m_7 \in \{-1, 1\}$.

first 6 qubits of the encoded state $|\bar{\psi}\rangle$ in Shor's code

ancilla qubit



• a similar circuit for measuring $X_4 X_5 X_6 X_7 X_8 X_9$: obtaining m_8

• Once we get all the measurement outcomes, we can now infer errors that most likely occurred and hence invert them to recover the encoded state (hopefully!)

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	most likely Error
+1	+1	+1	+1	+1	+1	+1	+1	$\mathbb{1}$, with probability $(1-p)^9$
-1	+1	+1	+1	+1	+1	+1	+1	X_1, with probability $p(1-p)^8$
⋮								
-1	+1	-1	+1	+1	-1	-1	-1	$Y_4 X_7$ with probability $p^2(1-p)^7$

there are also other choices, which are equivalent.

bit flip error in the second block: 4^{th} qubit

bit flip error in the 3^{rd} block: 7^{th} qubit

phase flip error in block 2

Tutorial: Ask students to try inferring the error from a few more measurement outcomes.

Correctable and Uncorrectable errors for the Shor Code:

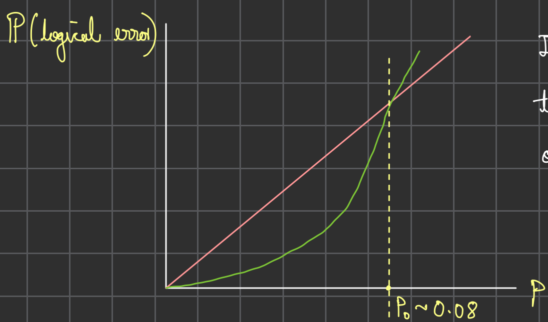
Correctable: All single qubit errors, 2 X errors in different blocks, 2 Z-errors within any block, ...

$$P(\text{correctable}) = (1-p)^3 + (9\times 3)\, p(1-p)^8 + \binom{3}{2} 3\, p^2 (1-p)^7 + 3 \cdot \binom{3}{2} p^2 (1-p)^7 + \dots$$

$$\text{Logical error} = 1 - P(\text{correctable})$$

$$\leq \underbrace{1 - (1-p)^3 - 27\,p(1-p)^8 - 18\,p^2(1-p)^7}_{\mathcal{O}(p^2)}$$

when is it beneficial to do QEC: logical error probability $\leq p$



If the Depolarizing channel's rate ≤ 0.08, then the Shor's code provides a suppression on the logical error.

• P_0 is often called the "breakeven point" or the pseudo-threshold.

• estimated using Monte-Carlo simulations.

Intuitively: By adding redundancy, we are adding more error-location. Hence if $p > p_0$, we are doing more bad than good.

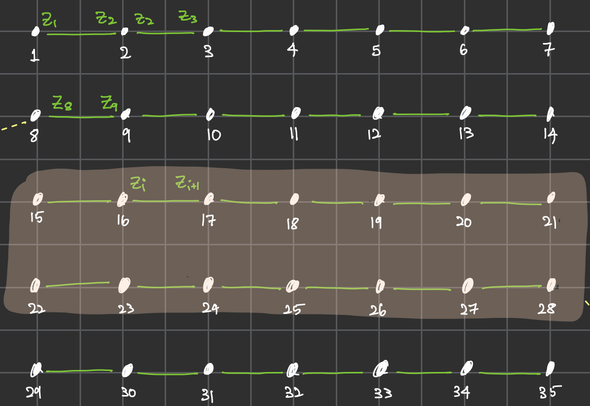
Further suppressing the logical rate : Quantum Parity Code

Generalizing the idea of correcting bit-flip and phase flip errors:

$$|\bar{0}\rangle = \frac{\left(|0\rangle^{\otimes m_1} + |1\rangle^{\otimes m_1}\right)^{\otimes m_2}}{(\sqrt{2})^{m_2}} \qquad |\bar{1}\rangle = \frac{\left(|0\rangle^{\otimes m_1} + |1\rangle^{\otimes m_1}\right)^{\otimes m_2}}{(\sqrt{2})^{m_2}}$$

- Note that in each of the blocks, we can correct $\left\lfloor \frac{m_1}{2} \right\rfloor - 1$ bit flip errors.
  - The operators that can be measured in this block are: $\{Z_i Z_{i+1} : i \in \text{block}\}$.
- We can correct $\left\lfloor \frac{m_2}{2} \right\rfloor - 1$ phase errors, by measuring $(X_i X_{i+1} \cdots X_{i+m_1}) \otimes (X_{i+m_1+1} \cdots X_{i+m_1+m_1})$

- Concise definition of these measurements:
  - Take an example: $m_1 = 7$, $m_2 = 5$.



qubits in the encoded state

- Horizontal links between qubits i and j denote that a $Z_i Z_j$ measurement can be done to detect X errors in the qubits i and j.

A pair of consecutive rows represent a collective $2m_1$-qubit X measurement : $X_i \ldots\ldots X_{i+m_1} \cdots X_{i+m_1+m_1}$, to detect any phase errors in the blocks $(i \ldots i+m_1)$ and $(i+m_1+1 \ldots i+2m_1)$.

Question to think about

- Remember we want to be able to correct arbitrary errors: E
- Take advantage of the expansion of $E = \underbrace{\alpha}_{\mathrm{Tr}(E)} I + \underbrace{\beta}_{\mathrm{Tr}(EX)} X + \gamma Y + \delta Z$. ...
- Explain that when we have $E|\psi\rangle = \alpha|\psi\rangle + \beta X|\psi\rangle + \gamma Y|\psi\rangle + \delta Z|\psi\rangle$, then when we measure one of the operators to detect errors, then:

$$\frac{(\mathbb{1} + Z_1 Z_2)}{2} E|\psi\rangle = \begin{cases} (\alpha|\psi\rangle + \delta Z|\psi\rangle) & \text{when the outcome} = 0 \\ (\beta X|\psi\rangle + \gamma Y|\psi\rangle) & \text{when the outcome} = 1 \end{cases}$$

- Likewise when we measure all the syndromes, we will finally end up with only one unique Pauli P acting on $|\psi\rangle$: $P|\psi\rangle$.
- Furthermore P is the unique single-qubit error consistent with measurement outcomes.
- Hence using measurement to project onto one Pauli Error, we can correct an arbitrary linear combination of Pauli Errors.

If we can correct errors from a set $\{E_i\}$, can the correction also work for other errors?

Error discretization theorem: If the errors in a channel $\mathcal{E}$ given by $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ are correctable then any channel $\mathcal{F}(\rho) = \sum_k F_k \rho F_k^\dagger$ given by: $F_k = \sum_j m_{kj} E_j$. by the same recovery operation.

Let us assume that $\{E_i\}$ are such that: $P E_i^\dagger E_j P = d_{ii} P$.

Applying the recovery $\mathcal{R}(\rho) = \sum_k U_k^\dagger P_k \rho P_k U_k$ to the state $\mathcal{F}(\rho)$ we find:

$$\mathcal{R}(\mathcal{F}(\rho)) = \sum_k \sum_i \underbrace{U_k^\dagger P_k^\dagger}_{\frac{(E_k P U_k^\dagger)^\dagger}{\sqrt{d_{kk}}}} E_i \, \rho \, E_i^\dagger \underbrace{P_k U_k}_{\frac{(E_k P U_k^\dagger)}{\sqrt{d_{kk}}}}$$

$$= \sum_k \sum_i \frac{U_k^\dagger U_k P E_k^\dagger E_i \, \rho \, E_i^\dagger E_k P U_k^\dagger U_k}{d_{kk}}$$

$$= \sum_{ki} \frac{P E_k^\dagger E_i (P \rho P) E_i^\dagger E_k P}{d_{kk}}$$

$$= \sum_{ki} \frac{(P E_k^\dagger E_i P) \rho (P E_i^\dagger E_k P)}{d_{kk}}$$

$$= \sum_{ki} \frac{\delta_{ki} d_{kk} d_{kk} P \rho P}{d_{kk}}$$

$$= d_{kk} \rho$$

Hence we see that if we can design a recovery for a set of errors, then we can apply the same recovery for any linear combination of errors.