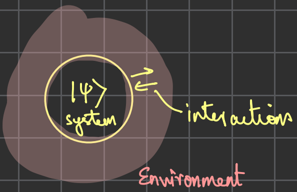
Monday, 15 April, 2024

Lecture 2: Challenges of QEC, Bit & Phase flip codes

References: Quantum Computation and Quantum Information by Neilsen & Chuang, Chapter 10.

Perimeter Institute Recorded Seminar Archive: PIRSA: https://pirsa.org/c17045

- Quantum systems constantly interact with their environment.



$\xrightarrow[\text{Effective dynamics}]{}$ Noise $\mathcal{E} : \rho \mapsto \mathcal{E}(\rho)$

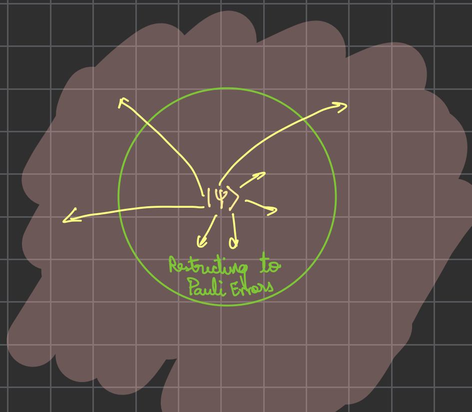
$\mathcal{E}$ can be any CPTP map, or an operation that preserves the properties of a density matrix.

- In any application we want to encode information into a state $|\psi\rangle$ and do meaningful operations reliably.
- Simple case: can we encode quantum information in $|\psi\rangle$ and preserve it for some time?
  - Case of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$: a single qubit (2-level system)

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow[\text{time}: t]{\mathcal{E}} |\phi\rangle \neq |\psi\rangle$$ ✗ Hopeless situation

Challenge #1: There are infinitely many possible errors

$\alpha|0\rangle + \beta|1\rangle$
- $\xrightarrow{\mathcal{E}_1} \alpha|1\rangle + \beta|0\rangle$
- $\xrightarrow{\mathcal{E}_2} \alpha|1\rangle - \beta|0\rangle$
- $\xrightarrow{\mathcal{E}_3} \alpha|0\rangle + \beta|0\rangle$
- $\xrightarrow{\mathcal{E}_4} \alpha|0\rangle + e^{i\theta}\beta|1\rangle$
- ⋮
- and so on



Out of all possible Errors we will restrict our attention to Pauli Matrices (Errors)

$$\mathcal{E}(\rho) = p_I\, \rho + p_X\, X \rho X + p_Y\, Y \rho Y + p_Z\, Z \rho Z. \qquad \text{(Pauli channel)}$$

$p_I + p_X + p_Y + p_Z = 1$: Trace preserving.

Recall that $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|\psi\rangle$ $(\alpha|0\rangle + \beta|1\rangle)$

- Prob : p_I — $|\psi\rangle$
- Prob: p_X — $X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$
- Prob: p_Y — $Y|\psi\rangle = \alpha|1\rangle + i\beta|0\rangle$
- Prob: p_Z — $Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$

Later on, in lecture #3, we will see that this is not as restrictive as it seems to be.

Suppose we have a state $|\psi\rangle$ that encodes quantum information, and

$$\underset{(\alpha|0\rangle+\beta|1\rangle)}{|\psi\rangle} \xrightarrow[\text{after time } T]{} \mathcal{E}(|\psi\rangle\langle\psi|)$$

can we recover the stored quantum information from $\mathcal{E}(|\psi\rangle\langle\psi|)$?

We want to encode the quantum information stored in $|\psi\rangle$:

(encode) $\quad U: |\psi\rangle \longmapsto |\phi\rangle$

such that there exists some operation $\mathcal{R}$ where:

$$|\phi\rangle \rightsquigarrow \mathcal{E}(|\phi\rangle\langle\phi|) \xrightarrow[\substack{\text{(quantum error} \\ \text{correction)}}]{\mathcal{R}} |\phi\rangle\langle\phi| \quad \text{with high probability.}$$

Toy Example : say we only care about protecting quantum information against X-errors:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Prob: $1-p$ — $\alpha|0\rangle + \beta|1\rangle$
- Prob: p — $\alpha|1\rangle + \beta|0\rangle$

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\, X \rho X \quad : \text{Bit flip channel}$$

Naive strategy: "playing the odds"

$$|\psi\rangle \; (\alpha|0\rangle + \beta|1\rangle) \xrightarrow[\text{(encode)}]{U} |\psi\rangle|\psi\rangle|\psi\rangle$$

$(1-p)^3 \, |\psi\rangle|\psi\rangle|\psi\rangle$

$p(1-p)^2 \, (X|\psi\rangle)\, |\psi\rangle|\psi\rangle$

$\vdots$

$p^2(1-p)(X|\psi\rangle)(X|\psi\rangle)\, |\psi\rangle$

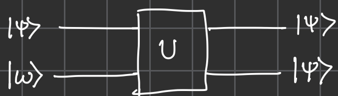
$\vdots$

$p^3 \, X|\psi\rangle \, X|\psi\rangle \, X|\psi\rangle$

- Look at the three copies, take a majority vote.
- If only one copy is affected, we will succeed.

- Obviously does not work : challenge #2

- No cloning theorem :



X not possible

- There exists no unitary operation U such that for any state $|\psi\rangle$:

$$U|\omega\rangle|\psi\rangle = |\psi\rangle|\psi\rangle$$

- Simple proof : Try cloning two states $|\psi\rangle$ and $|\phi\rangle$:

$$U(|\psi\rangle \otimes |\omega\rangle) = |\psi\rangle \otimes |\psi\rangle$$

and $$U(|\phi\rangle \otimes |\omega\rangle) = |\phi\rangle \otimes |\phi\rangle$$

Since U is a unitary operator, it should preserve the inner product:

$$\left.\begin{array}{l} U|x\rangle = |x'\rangle \\ U|y\rangle = |y'\rangle \end{array}\right\} \text{ then: } \langle x|y\rangle = \langle x'|y'\rangle.$$

Lets see what this reveals about states in our case:

$$(\langle\psi| \otimes \langle\omega|)\,(|\phi\rangle \otimes |\omega\rangle) = (\langle\psi| \otimes \langle\psi|)\,(|\phi\rangle \otimes |\phi\rangle)$$

$$\langle\psi|\phi\rangle\langle\omega|\omega\rangle = |\langle\psi|\phi\rangle|^2$$

So: $\langle\psi|\phi\rangle = |\langle\psi|\phi\rangle|^2$

implying that $\langle\psi|\phi\rangle = \begin{cases} 1 : & \text{meaning that } |\psi\rangle = |\phi\rangle \text{ and we can only clone a fixed state} \\ 0 : & \text{meaning we can only clone orthogonal states.} \end{cases}$

• What do we do to make the "play the odds" strategy work?

Clone only orthogonal states:

(encode) $U: |0\rangle \longmapsto |000\rangle$

$U: |1\rangle \longmapsto |111\rangle$

$$\alpha|0\rangle+\beta|1\rangle \xrightarrow[U]{\text{encode}} \alpha|000\rangle+\beta|111\rangle$$

$$|\psi\rangle = \alpha|0\rangle+\beta|1\rangle \xrightarrow[\text{(encode)}]{U} \alpha|000\rangle+\beta|111\rangle$$

$\xrightarrow{p}$ $\alpha|100\rangle+\beta|011\rangle = X_1|\psi\rangle$

$\xrightarrow{p}$ $\alpha|010\rangle+\beta|101\rangle = X_2|\psi\rangle$

$\xrightarrow{p}$ $\alpha|001\rangle+\beta|110\rangle = X_3|\psi\rangle$

$\xrightarrow{p^2}$ $\alpha|110\rangle+\beta|001\rangle = X_1X_2|\psi\rangle$

$\xrightarrow{p^2}$ $\alpha|101\rangle+\beta|010\rangle = X_1X_3|\psi\rangle$

$\xrightarrow{p^2}$ $\alpha|011\rangle+\beta|100\rangle = X_2X_3|\psi\rangle$

$\xrightarrow{p^3}$ $\alpha|111\rangle+\beta|000\rangle = X_1X_2X_3|\psi\rangle$

Can we measure each qubit in Z and then take a majority vote?

No: challenge #3: measurement collapses the quantum superposition.

$$\alpha|100\rangle+\beta|011\rangle \xrightarrow[Z_1]{\text{measure}} \begin{cases} |100\rangle : \text{outcome } -1 \\ |011\rangle : \text{outcome } 1 \end{cases}$$

• Instead we are only allowed to perform measurements that do not collapse the encoded state:

$$\alpha|0\rangle+\beta|1\rangle \xrightarrow[\text{(encode)}]{U} \alpha|000\rangle+\beta|111\rangle \xrightarrow{p} \underbrace{\alpha|100\rangle+\beta|011\rangle}$$

measure $\underbrace{Z_1Z_2}_{m_1}$ and $\underbrace{Z_2Z_3}_{m_2}$

$m_1=-1$: parity of the first two bits is odd.

$m_2=-1$: parity of the last two bits is odd.

m_1: outcome of measuring Z_1Z_2

m_2: outcome of measuring Z_2Z_3

$m_1=+1,\ m_2=+1$: Probability $(1-p)^2$ of no error.

$m_1=+1,\ m_2=-1$: Prob. p of X error on qubit 1.

$m_1=-1,\ m_2=+1$: Prob. p of X error on qubit 3.

$m_1=-1,\ m_2=-1$: Prob. p of X error on qubit 2.

In all cases measuring Z_1Z_2, Z_2Z_3 does not collapse the state.

Toy model: circumventing all the challenges of QEC: The Bit flip code

A bit flip code encodes $|0\rangle$ and $|1\rangle$ states using

$$U: |0\rangle \longmapsto |000\rangle$$

$$U: |1\rangle \longmapsto |111\rangle$$

Terminologies

1. U: encoder / encoding map
2. $|\bar{0}\rangle := |000\rangle$ and $|\bar{1}\rangle := |111\rangle$ are called encoded states
3. $\mathrm{Span}\{|\bar{0}\rangle, |\bar{1}\rangle\}$ is called the Quantum Error Correcting Code: In this case, the Bit flip code.
4. logical qubits k: 2^k = dimension of the code = # basis vectors

   The bit flip code encodes 1 logical qubit into 3 physical qubits.

Note: measuring $Z_1 Z_2$ and $Z_2 Z_3$ yields: $Z_1 Z_2 |\bar{x}\rangle = m_x |\bar{x}\rangle$, $x \in \{0,1\}$, $m_x = +1$.

Environment: Bit flip channel

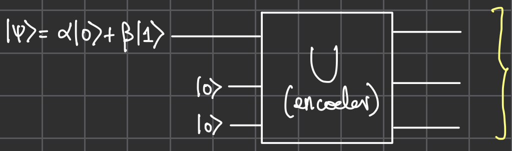
$\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$

$m_1 = 1$, $m_2 = 1$

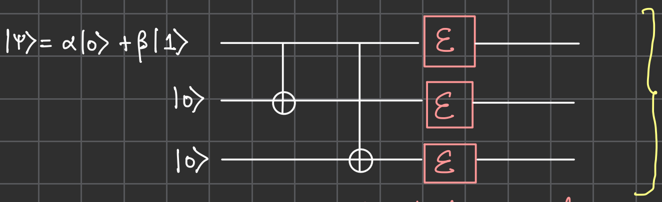
- $\alpha|100\rangle + \beta|011\rangle$ — $m_1 = -1$, $m_2 = +1$; X_1 error with prob. $p(1-p)^2$
- $\alpha|101\rangle + \beta|010\rangle$ — $m_1 = -1$, $m_2 = -1$; X_2 error with prob. $p(1-p)^2$
- $\alpha|001\rangle + \beta|110\rangle$ — $m_1 = +1$, $m_2 = -1$; X_3 error with prob. $p(1-p)^2$
- $\alpha|110\rangle + \beta|001\rangle$ — $m_1 = +1$, $m_2 = -1$; X_3 error with prob. $p(1-p)^2$ ✗
- ⋮
- $\alpha|111\rangle + \beta|000\rangle$ — $m_1 = +1$, $m_2 = +1$; No error ✗

Takeaway:
- We will be able to correct for any single qubit X error in the bit flip channel.
- Bit flip code fails to correct multiqubit X errors.

Circuit model for the Bit-Flip code:



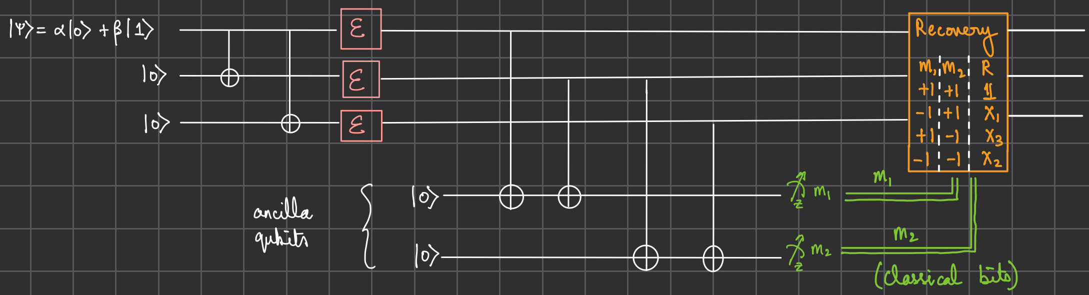
$$|\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle = \alpha|000\rangle + \beta|111\rangle$$



Measure Z_1Z_2, Z_2Z_3 to find m_1 and m_2

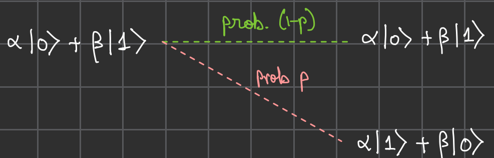
$\mathcal{E}$: Bit flip channel with probability

Doing measurements of Z_1Z_2 and Z_2Z_3 to find out information about the errors:

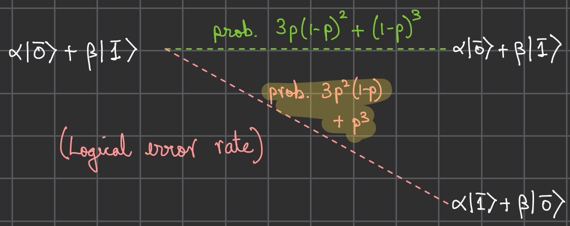


Summary: under bit flip channel, what is the net benefit of using the bit flip quantum error correction scheme?

Without QEC with the Bit-flip code



With QEC using the Bit flip code

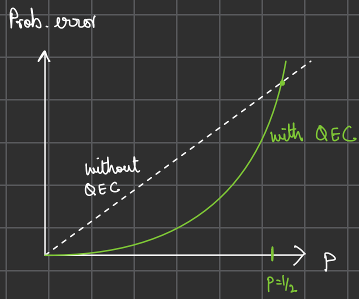


Probability of error:

$$p \quad \text{vs.} \quad 3p^2(1-p) + (1-p)^3$$

$$p \quad \text{v.s} \quad \mathcal{O}(p^2)$$

- As long as $p < 1/2$, we can suppress logical errors by employing QEC.



Tutorial topic:

What if we want a greater suppression on the logical error rate?

- Can we achieve a logical error rate of $\mathcal{O}(p^3)$, $\mathcal{O}(p^4)$, ...?
- Say we want a logical error rate of $\mathcal{O}(p^3)$: we want to correct all 2-qubit errors.

$$|0\rangle \mapsto |\bar{0}\rangle = \frac{|00000\rangle + |11111\rangle}{\sqrt{2}}$$

$$|1\rangle \mapsto |\bar{1}\rangle = \frac{|00000\rangle - |11111\rangle}{\sqrt{2}}$$

We are encoding one logical qubit in 5 physical qubits.

measure operators: Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_5

m_1	m_2	m_3	m_4	
+1	+1	+1	+1	: No error with prob. $(1-p)^5$
-1	+1	+1	+1	: X_1 error with prob. $p(1-p)^4$
	⋮			
-1	-1	+1	-1	: X_2X_5 error with prob. $p^2(1-p)^3$

Phase-Flip code:

- Recall that there are Z-errors too! What happens when a Z-error affects the states encoded in the bit flip code?

Phase flip channel: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ — Prob. $(1-p)$ → $\alpha|0\rangle + \beta|1\rangle$; Prob. p → $Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$

Bit flip code:

$$|\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle = \alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{on any qubit}]{Z \text{ error}} \alpha|000\rangle - \beta|111\rangle \qquad m_1 = +1,\ m_2 = +1$$

The measurements do not reveal anything useful: Bit flip code cannot correct errors in the phase flip channel.

Correcting Z-errors: the phase-flip code:

- Intuition: we knew that Z_1Z_2, Z_2Z_3 measurements are sensitive to the presence of X errors. So, the code was eigenstates of Z_1Z_2, Z_2Z_3. Now we want to be sensitive to the presence of Z errors, so the code should be eigenstates of X_1X_2, X_2X_3.

$$U: \begin{rcases} |0\rangle \longmapsto |\bar{0}\rangle := |+++\rangle \\ |1\rangle \longmapsto |\bar{1}\rangle := |---\rangle \end{rcases} \quad U: |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longmapsto |\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle = \alpha|000\rangle + \beta|111\rangle$$

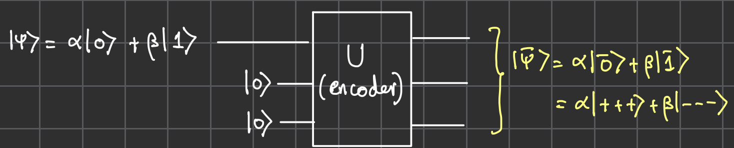
- span$\{|\bar{0}\rangle = |+++\rangle, |\bar{1}\rangle = |---\rangle\}$ is the phase-flip code and U is its encoder.
- measuring X_1X_2 and X_2X_3 on $|\bar{\psi}\rangle$: $X_1X_2|\bar{\psi}\rangle = m_1|\bar{\psi}\rangle$, $X_2X_3|\bar{\psi}\rangle = m_2|\bar{\psi}\rangle$ where $m_1 = m_2 = +1$

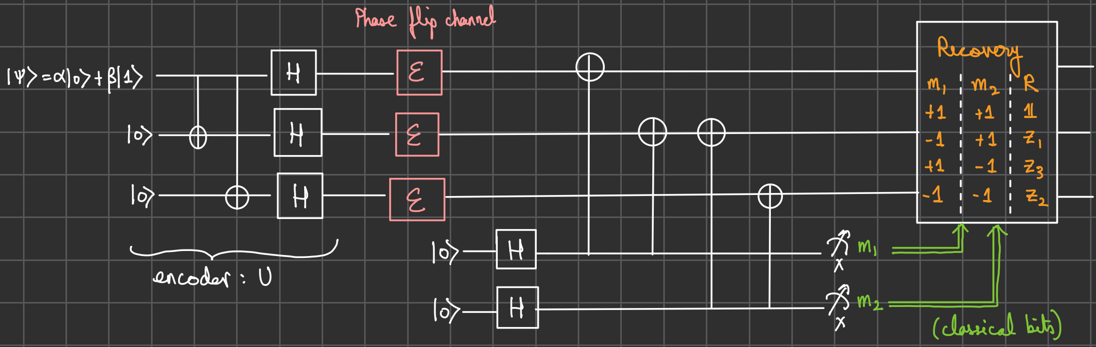
$\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle = \alpha|+++\rangle + \beta|---\rangle$ $\longrightarrow$

- $\alpha|-++\rangle + \beta|+--\rangle$ — $m_1 = -1$, $m_2 = +1$: Z_1 error with prob. p
- $\alpha|+-+\rangle + \beta|-+-\rangle$ — $m_1 = -1$, $m_2 = -1$: Z_2 error with prob. p
- $\alpha|++-\rangle + \beta|--+\rangle$ — $m_1 = +1$, $m_2 = -1$: Z_3 error with prob. p
- $\alpha|--+\rangle + \beta|++-\rangle$ — $m_1 = +1$, $m_2 = -1$ ✗: Z_3 error with prob. p.
- $\vdots$

- Phase flip code can correct any single qubit Z-error on the phase flip channel.

Circuit model for the phase flip code:





Hence the phase flip code protects against single qubit errors in the phase flip channel.

	Bit flip channel	Phase flip channel	Any single qubit error
Bit flip code	✓	✗	✗
Phase flip code	✗	✓	✗

- Can we find a code that corrects bit flip as well as phase flip channels?

Current Research :
- Plenty of QEC codes at : errorcorrectionzoo.org (QEC Zoo)
- Experimental demonstration of Phase-flip code with semiconductor qubits
  npj Quantum Information, 8, Article No. 124, 2022.
- QEC with Silicon Spin qubits
  Nature 608, 682-686 (2022)