York time in JT gravity

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Introduction

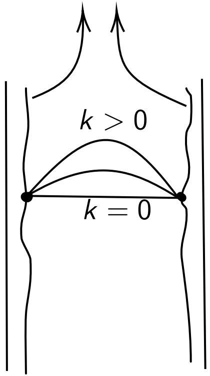
- In general relativity, the notion of time is not an external parameter, but must be specified intrinsically in terms of the spacetime metric.
- When a time-like asymtotic boundary is available, there is a clear notion of boundary time, which one may regard as an "external" clock.
- The corresponding Hamiltonian, i.e., the ADM Hamiltonian, ends up being a boundary term.
- In AdS/CFT, the ADM Hamiltonian is the bulk avatar of the microscopic Hamiltonian of the dual quantum mechanical system.

Introduction

- However, in spacetimes like de Sitter with no timelike boundaries, one does not have recourse to a boundary notion of time and one is forced to consider an "internal" notion of time.
- This is one of the features of de Sitter space which obfuscates a microscopic understanding.
- In this talk, we will consider an internal notion of time in asymptotically AdS spacetimes, where a dual quantum mechanical description *is* available.
- Our ultimate goal is to develop some sort of microscopic understanding of internal time evolution – with the hope that the lessons can then be generalized.

Introduction

- Consider foliating the bulk spacetime with slices anchored at the same boundary times but with different values of the extrinsic curvature – this notion of time is called the *York time* [York '71, Witten '22].



- It is interesting to consider whether this notion of evolution might also find a microscopic realization in the dual boundary theory.

Introduction

- We will focus on the simple setting of JT gravity in AdS_2, where the boundary dual is given by random matrix theory [SSS '19].
- At a technical level, we will obtain a slight generalization of the Hartle-Hawking (HH) wavefunction of JT gravity.
- The standard HH wavefunction can be interpreted as computing the overlap of the thermo-field double state in the boundary quantum theory [Maldacena '01] with certain special fixed-geodesic-length states [Harlow & Jafferis '18, Yang '18].
- From the perspective of York time, we can think of this as the wavefunction at $k = 0$, i.e., at zero York time.
- We wish to study the York time evolution of the HH wavefunction.

JT gravity

- JT gravity is a two-dimensional model of gravity with the action [Maldacena, Stanford, Yang '16]:

$$S_{\rm JT} = S_{\rm bulk} + S_{\rm bdy},$$
$$S_{\rm bulk} = -\frac{1}{16\pi}\int_M d^2x\sqrt{g}\,\phi\,(R+2),$$
$$S_{\rm bdy} = -\frac{1}{8\pi}\int_{\partial M} dx\,\sqrt{\gamma}\,\phi\,K,$$

  where ϕ is the dilaton, and R is the Ricci scalar of the two dimensional spacetime.
- The dilaton and the induced metric are held fixed on the asymptotic boundary:

$$\phi_{\partial M} = \frac{1}{\epsilon}\phi_b, \quad \gamma = \frac{1}{\epsilon^2}d\tau^2.$$

JT gravity

- The equations of motion for JT gravity are given by:

$$R + 2 = 0,$$
$$\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi - g_{\mu\nu} \phi = 0.$$

- The first of these equations fixes the metric to be AdS_2 everywhere:

$$ds^2 = (r^2 - r_s^2) d\tau^2 + \frac{dr^2}{(r^2 - r_s^2)}.$$

- A solution for ϕ, which is independent of τ, is given by

$$\phi = \phi_b r.$$

- In these coordinates, the boundary is at $r = \epsilon$.

Phase space of JT gravity : first look

- In Lorentz signature, we can always write the metric as:

$$ds^2 = N^2 dt^2 + h(dx + Mdt)(dx + Mdt),$$

where N and M are the lapse and shift functions, while $\mathfrak{h} = h\,dx^2$ is the induced metric on a Cauchy slice.

- Standard canonical analysis shows [Henneaux]

$$\pi_\phi \equiv \frac{\delta L}{\delta \dot{\phi}} = -\sqrt{h}\,K,$$
$$\pi_h \equiv \frac{\delta L}{\delta \dot{h}} = -\frac{1}{2\sqrt{h}} \mathfrak{n}^\alpha \nabla_\alpha \phi,$$

where K is the extrinsic curvature of the Cauchy slice and $\mathfrak{n}$ is the unit normal to the Cauchy slice.

- Naively, the phase space consists of $(\phi, \pi_\phi, h, \pi_h)$, all of which are functions of x.

Phase space of JT gravity: first look

- The variables N and M are Lagrange multipliers and enforce the Gauss law constraints of JT gravity:

$$\sqrt{h}(-\phi - D^2\phi + K \mathfrak{t}^\alpha \nabla_\alpha \phi) = 0,$$
$$\sqrt{h}\,(K\partial_x\phi - \partial_x(\mathfrak{t}^\alpha \nabla_\alpha \phi)) = 0.$$

- We can choose to foliate spacetime with slices of constant extrinsic curvature between two fixed end points on the asymptotic boundary, and use the value of the extrinsic curvature as our time coordinate.
- Furthermore, we can choose the spatial coordinate to be the proper length along these slices to set $h = 1$.

Phase space of JT gravity: first look

- The only physical information in the induced metric, then, is the (renormalized) length of the slice:

$$\ell_{\text{ren.}} = \int dx \sqrt{h} - \text{div.}$$

- The Gauss law constraints allow us to solve for the dilaton and its normal derivative, up to a constant of integration.
- This constant is related to the canonical momentum P of the length of constant time slices.
- Thus, the physical phase space of JT gravity is two-dimensional $(\ell_{\text{ren.}}, P)$.

Phase space of JT gravity: covariant approach

- A covariant way of thinking about the phase space is as the set of the solutions:

$$ds^2 = -(1+x^2)dt^2 + \frac{dx^2}{1+x^2}, \quad \phi = \phi_h \sqrt{1+x^2}\cos(t),$$

  which are just Kruskal extensions of the black holes we encountered previously [Harlow, Jafferis '18].
- Here $\phi_h = r_s \phi_b$ is a parameter labeling the space of solutions.
- But phase space must be even dimensional – the other parameter is obtained by evolving the solutions by Schwarzschild boundary time δ on both sides.
- We can think of this as complexifying the temperature.

Phase space of JT gravity

- This gives the system of equations

$$\dot{\delta} = 1, \dot{\phi}_h = 0$$

- The ADM Hamiltonian for the right boundary and the symplectic form on phase space are given by

$$H = H_L + H_R = 2\frac{\phi_h^2}{\phi_b}, \;\; \omega = d\delta \wedge dH.$$

- This discussion makes it physically clear that the phase space of solutions is two-dimensional.

Going back to the length variable

- We would now like to better understand the phase space in terms of the length variable.
- In AdS_2, there always exists a unique codimension-1 spacelike slice with constant extrinsic curvature k and ending on specified boundary end points:

$$x(\lambda) = \frac{\sinh\left(\sqrt{k^2+1}\,\lambda\right)}{\sqrt{k^2+1}},$$
$$\tau(\lambda) = \tau_0 + \tan^{-1}\left(k\,\mathrm{sech}\left(\lambda\sqrt{k^2+1}\right)\right),$$

- There are three constants here: the values of the affine coordinate λ at the two end points, and the constant τ_0.
- Correspondingly, there are three constraints: that the dilaton values at the two end points should be ϕ_b, and the Schwarzschild times should be $(t_L, t_R) = (\delta, \delta)$.

Going back to the length variable

- We can compute the length for the above slice. After suitable renormalization, we get

$$\ell_{\text{ren.}}(k) = \ell(k) - \frac{2}{\sqrt{1+k^2}} \log(\frac{\sqrt{1+k^2}}{\epsilon})$$
$$= \frac{2}{\sqrt{1+k^2}} \log\left(\frac{2\phi_b}{\phi_h} \cosh\left(\frac{\phi_h}{\phi_b}\delta\right)\right).$$

- Requiring that the symplectic form should be given by

$$\omega = d\ell_{\text{ren.}} \wedge dP,$$

we can find the corresponding momentum:

$$P(k) = \phi_h \sqrt{1+k^2} \tanh\left(\frac{\phi_h \delta}{\phi_b}\right).$$

Going back to the length variable

- The ADM Hamiltonian in terms of these variables can be written as

$$H_{\mathrm{ADM}}(k) = \frac{P^2}{\phi_b(1+k^2)} + 4\phi_b e^{-\ell_{\mathrm{ren.}}\sqrt{1+k^2}}.$$

- To summarize, we have managed to write the ADM Hamiltonian in terms of the length and its conjugate momentum at any specified value of York time k.
- Our goal now is to promote this Hamiltonian to a quantum operator and use it to compute the Hartle-Hawking wavefunction at York time k.

The Hartle-Hawking wavefunction

- In going to the quantum theory, we encounter operator-ordering issues.
- For example, writing $\widehat{\ell}_{\text{ren.}}(k)$ or $\widehat{P}(k)$ in terms of $\widehat{\phi}_h$ and $\widehat{\delta}$ is complicated by operator-ordering ambiguities.
- On the other hand, there are no ambiguities in writing $\widehat{H}_{\text{ADM}}$ in terms of $\widehat{\ell}_{\text{ren.}}(k)$ and $\widehat{P}(k)$, so this is the prescription we follow:

$$\widehat{H}_{\text{ADM}} = \frac{\widehat{P}(k)^2}{\phi_b(1+k^2)} + 4\phi_b e^{-\widehat{\ell}_{\text{ren.}}(k)\sqrt{1+k^2}}.$$

The Hartle-Hawking wavefunction

- Let $|\ell, k\rangle$ be states with fixed renormalized length at extrinsic curvature k:
$$\widehat{\ell}_{\text{ren.}}(k)|\ell, k\rangle = \ell|\ell, k\rangle.$$
- We wish to compute the overlap $\langle \ell, k|E\rangle$, where E is an eigenstate of the ADM Hamiltonian.
- Using the explicit form of the ADM Hamiltonian in the equation:
$$\langle \ell, k|\widehat{H}_{ADM}|E\rangle = E\,\langle \ell, k|E\rangle,$$
gives us a differential equation:

$$\left(-\frac{1}{\phi_b(1+k^2)}\partial_\ell^2 + 4\phi_b e^{-\ell\sqrt{1+k^2}}\right)\langle \ell, k|E\rangle = E\langle \ell, k|E\rangle.$$

The Hartle-Hawking wavefunction

- The (delta function) normalizable solution is given by

$$\langle \ell, k | E \rangle = K_{2i\sqrt{E\phi_b}}\left(4\phi_b e^{-\frac{1}{2}\ell\sqrt{1+k^2}}\right).$$

- The Hartle-Hawking wavefunction in the length basis at York time k can now be obtained as

$$\Psi_{HH}(\ell, k) = \langle \ell, k | \mathrm{TFD}_\beta \rangle = \int_0^\infty dE\, \rho(E) e^{-\frac{\beta}{2}E} \langle \ell, k | E \rangle.$$

$$\rho(E) = \frac{2}{\pi^2}\sinh(2\pi\sqrt{E}).$$

Overlap of length eigenstates

- We can compute the norm of the states $|\ell, k\rangle$ by noting that the energy eigenstates have the following completeness relations

$$\int_0^\infty dE\, \rho(E)\, |E\rangle\langle E| = \mathbb{1},$$

and we find

$$\begin{aligned}\langle \ell_1, k|\ell_2, k\rangle &= \int_0^\infty dE\, \rho(E)\, \langle \ell_1, k|E\rangle\langle E|\ell_2, k\rangle \\ &= \frac{1}{\sqrt{1+k^2}}\delta(\ell_1 - \ell_2).\end{aligned}$$

- Note the time-dependent factor in the normalization.

York time evolution

- It is now easy to check that the HH wavefunction (or its microcanonical counterpart) satisfies the following Schrodinger equation:

$$-i\partial_k \langle \ell, k | E \rangle = \langle \ell, k | \widehat{H}_{\text{York}} | E \rangle$$

where

$$\widehat{H}_{\text{York}}(k) = \frac{k}{1+k^2} \widehat{\ell}(k)\, \widehat{P}(k).$$

- Naively, the York Hamiltonian *does not* look like a Hermitian operator.
- However, this is because the length eigenstates are not properly normalized.

York time evolution

- We define the new basis states:

$$|\ell, k\rangle_{\text{new}} = (1+k^2)^{\frac{1}{4}} |\ell, k\rangle_{\text{old}}.$$

- The overlap of these new basis states with energy eigenstates is given by

$$\langle \ell, k|E\rangle_{\text{new}} = (1+k^2)^{\frac{1}{4}} \langle \ell, k|E\rangle_{\text{old}}.$$

- Because of the extra k-dependence in the prefactor, the new wavefunction satisfies the same Schrodinger equation:

$$-i\partial_k \langle \ell, k|E\rangle = \langle \ell, k|\widehat{H}_{\text{York}}|E\rangle$$

but now with the modified Hamiltonian:

$$\widehat{H}_{\text{York}}(k) = \frac{1}{2}\frac{k}{1+k^2}\Big(\widehat{\ell}(k)\,\widehat{P}(k) + \widehat{P}(k)\,\widehat{\ell}(k)\Big),$$

which is now manifestly Hermitian.

Why time evolution?

Question: bulk time evolution (at fixed boundary time) should leave the boundary state unchanged. So then why does the HH wavefunction satisfy the Schrodinger equation for York time evolution?

Relational observables

- Our picture is that the York time dependence of the HH wavefunction comes from a k-dependent unitary transformation of the length operator and the corresponding fixed-length basis states:

$$\widehat{\ell}(k) = U(k)\,\widehat{\ell}(0)\,U^\dagger(k), \quad |\ell, k\rangle = U(k)\,|\ell, 0\rangle,$$

$$U(k) = \mathcal{T}\exp\left[i\int_0^k dk'\,\widehat{H}_{\text{York}}(k')\right]. \qquad (1)$$

- In fact, this unitary is simply given by the dilatation operator:

$$U(k) = e^{i\frac{\kappa}{2}\left(\widehat{\ell}\,\widehat{P}+\widehat{P}\,\widehat{\ell}\right)}, \quad \kappa = \log(1+k^2)^{1/4},$$

where $\widehat{\ell}$ and $\widehat{P}$ are geodesic operators (i.e., at $k=0$).

Relational observables

- Thus, the HH wavefunction in this basis satisfies a Schrodinger equation, not because the physical state $|E\rangle$ or $|\mathrm{TFD}_\beta\rangle$ is evolving, but because the basis we use to compute the wavefunction is changing unitarily.
- This happens because we are using basis states which diagonalize a new observable, namely the length at York time k.
- We expect that our picture is very closely related to the notion of "relational observables" [Page & Wooters, Hoehn et al '19...]. It would be interesting to develop this connection further.

Outlook

- Our analysis suggests that the emergence of bulk time is tied with unitary rotations of the geometric basis in gravity.
- The really interesting, and important question is to understand this bulk notion of time evolution from a microscopic point of view.
- For this, we need a microscopic understanding of the geodesic length basis of gravity.
- There are some recent proposals. For instance, the Krylov basis in the boundary theory has emerged as an interesting candidate [Balasubramanian et al '21, Lin '22, Sonner '23, Nandy '24...], but there are also other proposals [Iliesiu, Levine, Lin, Maxfield, Mezei '24].

Outlook

- The Krylov basis is a discrete basis, analogous to the chord states in DSSYK:

$$\mathcal{K} = \left\{ |\Omega\rangle, H|\Omega\rangle, H^2|\Omega\rangle \cdots \right\}_{GS}$$

- In the double-scaling limit, it approaches the continuous length basis of JT gravity [Berkooz et al '18, Lin '22, Okuyama].
- But what is the analog of the unitary transformation (dilatation) that we found in JT gravity?

Outlook: a discrete phase space formalism

- In quantum information theory, people have constructed a discrete phase space for finite dimensional Hilbert spaces [Wooters '87, Gibbons, Hoffman & Wooters '04, Gross '06 ...].
- For a Hilbert space of dimension D, the phase space is taken to be the lattice $\mathcal{P} = \mathbb{Z}_D \times \mathbb{Z}_D$.
- The formalism works best when D is an odd prime integer, but can be generalized to arbitrary D.

Outlook: a discrete phase space formalism

- Let $\{|q\rangle\}_{q=0}^{D-1}$ be a choice of an *ordered*, orthonormal basis.
- With respect to this basis, we define exponentiated position and momentum operators as:

$$\widehat{Z}(p)|q\rangle = e^{\frac{2\pi i p q}{D}}|q\rangle, \quad \widehat{X}(q)|q'\rangle = |(q'+q) \bmod D\rangle.$$

- In fact, one can even construct a discrete Wigner function in direct analogy with the continuous case:

$$W(q,p) = \frac{1}{D}\sum_{q',q''=0}^{D-1} \tilde{\delta}_{2q,q'+q''} e^{-\frac{2\pi i (q'-q'')p}{D}} \langle\psi|q'\rangle\langle q''|\psi\rangle.$$

Outlook: Clifford unitaries

- The discrete Wigner function is a tool which allows us to visualize quantum states in a phase-space language.
- The negativity of the Wigner function gives an operationally meaningful measure of the complexity of classically simulating quantum states [Basu, Ganguly, Nath, OP '24].
- In this language, there is a natural group of unitaries called *Clifford unitaries*, which act "covariantly" on the Wigner function and leave its negativity invariant.
- These "classicality/complexity-preserving" transformations might serve as a natural candidate for the microscopic realization of York time evolution.

Outlook

- In AdS/CFT, we often talk about the emergence of a bulk spatial direction; in some sense, here we were interested in the emergence of bulk *time* from the boundary theory.
- Similar in spirit to "Cauchy slice holography" [Caputa, Kruthoff, OP '20, Araujo-Rigado, Khan, Wall '22].
- One key difference is that in Cauchy slice holography, one imposes Dirichlet boundary conditions on the Cauchy slices, where as in our case, York time slicing corresponds to part Neumann, part Dirichlet boundary conditions.
- From our point of view, the emergence of York time corresponds to a unitary transformation of the gravitational length basis states.
- In Cauchy slice holography, the flow corresponds to a $T\bar{T}$-deformation of the boundary Euclidean path integral.