Quantum Fields on the Maximally Sliced two-sided BTZ black hole

2501.03926 + wip with Naveen S. Prabhakar and Spenta R. Wadia

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Introduction

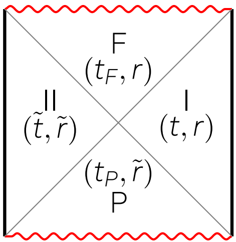
- In the second part, we discuss the dynamics of a probe scalar field in the background of the maximally sliced BTZ black hole solution.
- In the semi-classical expansion in $\kappa = \sqrt{16\pi G_N}$, the scalar field is gauge invariant at the quadratic order as it does not mix with the metric fluctuations.
- We write down the scalar field Hamiltonian which propagates along maximal slices in terms of a discrete set of mode functions that are smooth across the horizons. These modes mix the boundary operators from the two sides and are frequency smeared combinations of the Hartle-Hawking modes.
- This Hamiltonian is a finite, Hermitian and gauge-invariant operator in the product of the two CFTs associated to the two boundaries and describes the unitary time evolution of states / operators as they traverse the horizon.
- We also present two-point function calculation in the Hartle-Hawking vacuum in the 'wormhole' coordinates. Not all correlation functions decay.

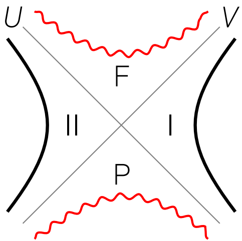
The BTZ black hole

- The BTZ black hole metric is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\varphi^2\,, \qquad f(r) = \frac{r^2}{\ell^2} - \frac{M}{2\pi}\,.$$

This metric has a coordinate singularity at the horizon $R_h = \ell\sqrt{\frac{M}{2\pi}}$.





- Recall that the BTZ black hole has a Kruskal extension which consists of four BTZ-like regions with their own local BTZ-like coordinates.
- Penrose diagram on the left and Kruskal diagram on the right.
- Thick black lines are the AAdS boundaries, the red wavy lines are the singularities and the thin gray lines are the horizons.

The wormhole coordinates

- We trade the θ coordinate for wormhole coordinate x. The maximally sliced BTZ black hole in wormhole coordinates $(\bar{t}, x, \varphi)$ is described by

$$ds^2 = -N^2 d\bar{t}^2 + g_{xx}\left(dx + N^x d\bar{t}\right)^2 + \left(x^2 + R_+(\bar{t})^2\right) d\varphi^2,$$

$$N(\bar{t}, x) = \frac{x\sqrt{x^2 + R_+^2 - R_-^2}}{\ell\sqrt{x^2 + R_+^2}}\left[1 + \dot{T}\ell^3 \int_x^\infty \frac{dy}{y^2}\frac{\sqrt{y^2 + R_+^2}}{\left(y^2 + R_+^2 - R_-^2\right)^{3/2}}\right],$$

$$N^x(\bar{t}, x) = \frac{1}{x}\left(N(\bar{t}, x)T(\bar{t}) + R_+\dot{R}_+\right), \qquad g_{xx}(\bar{t}, x) = \frac{\ell^2}{x^2 + R_+^2 - R_-^2},$$

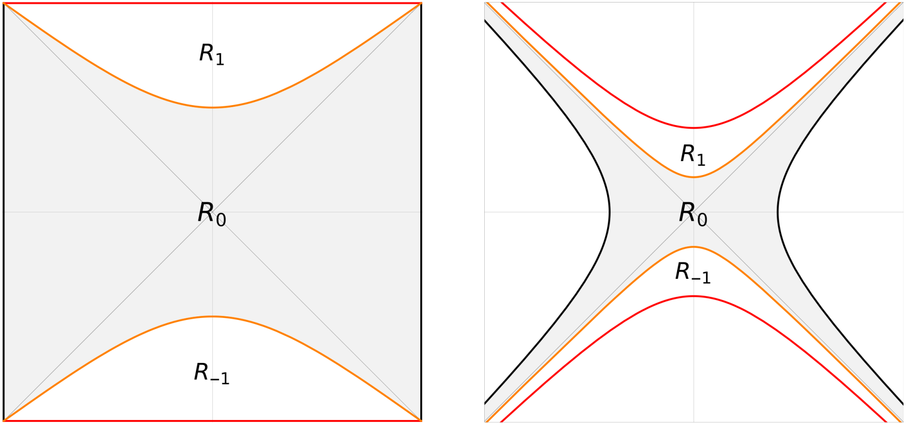
$$\bar{t} = -T(\bar{t})\ell^3 \int_0^\infty \frac{dy}{(y^2 - R_-^2)\sqrt{(y^2 + R_+^2)(y^2 + R_+^2 - R_-^2)}},$$

$$R_\pm(T) = \ell\sqrt{\frac{M}{4\pi}}\left[1 \pm \sqrt{1 - \left(\frac{4\pi T(\bar{t})}{M\ell}\right)^2}\right]^{1/2}.$$

Note: $-\infty < \bar{t} < \infty\,,\ -\infty < x < \infty\,,\ 0 \le \varphi \le 2\pi.$

Wormhole chart in the Kruskal extension of the BTZ BH

- The maximally sliced solution maps to the region R_0 in the full Kruskal extension of the BTZ black hole, shown in the Penrose diagram (left) and Kruskal diagram (right).



- The region R_0 contains both AAdS boundaries but excludes the singularities. A kind of 'cosmic censorship'.

Properties of BTZ black hole in wormhole coordinates

- Maximal slices extend from the left asymptotic boundary ($x = -\infty$) to the right asymptotic boundary ($x = +\infty$) while smoothly cutting across the horizons. The x coordinate runs along the length of the wormhole slices.

- At asymptotic boundaries $\bar{t}$ is the same as the boundary time t and $-\tilde{t}$.

- As $\bar{t} \to \pm\infty$ the foliation settles to a 'final slice' at $R_\infty = \ell\sqrt{\frac{M}{4\pi}}$, avoiding regions near the singularity (useful in numerical relativity).

- *Collapse of the lapse*: the lapse N remains finite and positive everywhere but vanishes as $N(\bar{t} \gg \eta^{-1}, x) \approx N_0(x)\mathrm{e}^{-\sqrt{2}\eta\bar{t}}$ when $\bar{t} \to \infty$.

- The BTZ coordinates t, r are related to the wormhole coordinates (for $\bar{t} \geq 0$)

$$r^2 = x^2 + R_+^2(\bar{t}), \qquad t = -T(\bar{t})\ell^3 \int_0^x \frac{\mathrm{d}y}{\left(y^2 - R_-^2(\bar{t})\right)\sqrt{\left(y^2 + R_+^2(\bar{t})\right)\left(y^2 + R_+^2(\bar{t}) - R_-^2(\bar{t})\right)}}.$$

Maximal slices

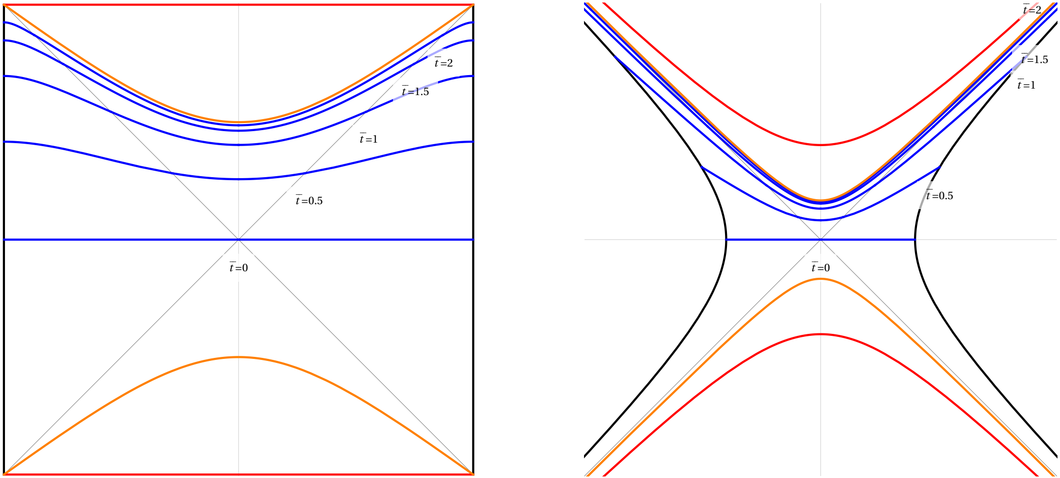


Figure: Constant $\bar{t}$ maximal slices are plotted as solid blue lines in the Penrose diagram (left) and Kruskal diagram (right).

Maximal slices

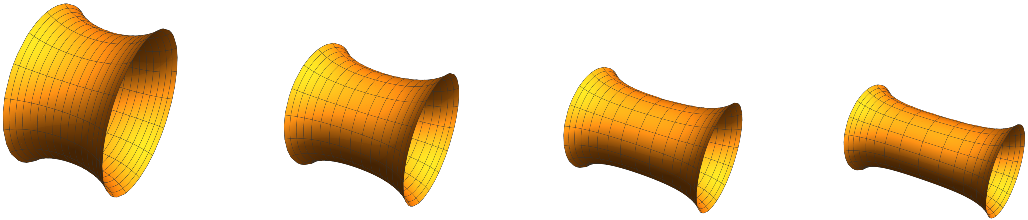


Figure: We plot the constant $\bar{t}$ slices between the two null horizons, for various values of $\bar{t}$. The φ coordinate runs along the circle direction while the x coordinate runs along the length of the wormhole. As $\bar{t}$ increases, the spatial wormhole stretches along the length and shrinks along the circle.

A second family of Maximal slices

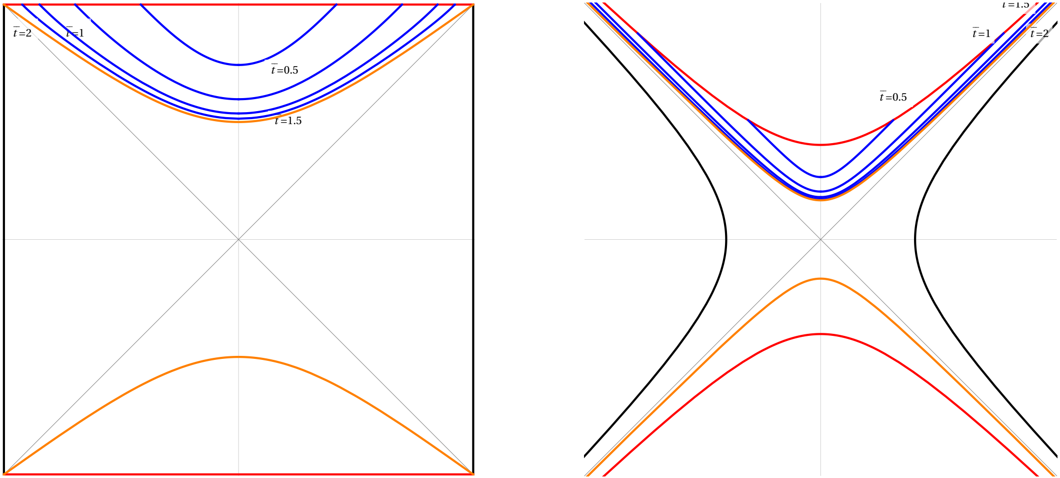


Figure: Constant $\bar{t}$ maximal slices of the 2nd family which stay inside the horizon, are plotted in the Penrose diagram (left) and Kruskal diagram (right). These slices start at the singularity, grow outwards and end at the final slice at $R_\infty = \ell\sqrt{M/4\pi}$ as $\bar{t} \to \infty$.

Probe scalar field in maximally sliced BTZ geometry

- At quadratic order in perturbation theory, the scalar field is gauge invariant and a physical degree of freedom.
- We would like to quantize the scalar field along maximal slices.
- KG equation in wormhole coordinates is difficult to solve directly due to the time dependence of the coefficients in the differential equation:

$$
\begin{aligned}
-\partial_{\bar{t}}\left(N^{-1}\sqrt{g}(\partial_{\bar{t}}\phi - N^i\partial_i\phi)\right) + \partial_i\left(N^{-1}\sqrt{g}N^i\partial_{\bar{t}}\phi\right) \\
+ \partial_i\left(N\sqrt{g}(g^{ij} - N^{-2}N^iN^j)\partial_j\phi\right) - N\sqrt{g}m^2\phi = 0.
\end{aligned}
$$

- Note that, given Cauchy initial data for the scalar field, there is no problem in solving this numerically since everything is smooth and well-defined.
- Strategy: solve the scalar field in the BTZ coordinates and pull it back onto the maximal slices.

Semi-classical perturbation theory

- We study gravity + scalar field in semi-classical expansion
$$S = \int \mathrm{d}t \int \mathrm{d}^d x \left(\pi^{ij}\dot{g}_{ij} - N\mathcal{H}_\perp - N_i\mathcal{H}^i\right) - \int \mathrm{d}t \int \mathrm{d}^{d-1}\sigma\mathcal{B}.$$
- Expand everything in $\kappa = \sqrt{16\pi G_N}$ around a solution of Einstein's equations:
$$g_{ij} = \hat{g}_{ij} + \kappa h_{ij}, \quad P^{ij} := \kappa^2\pi^{ij} = \hat{P}^{ij} + \kappa p^{ij}, \quad N = \hat{N}(1+\kappa\alpha), \quad N_i = \hat{N}_i + \kappa\beta_i,$$
$$S = \hat{S} + \kappa S_1 + \frac{1}{2}\kappa^2 S_2 + \cdots$$
- The Hamiltonian at quadratic order is
$$H_2 = \int \mathrm{d}^d x \left(\alpha C_\perp(h_{ij}, p^{ij}) + \beta_i C^i(h_{ij}, p^{ij}) + \mathcal{H}_{grav}(h_{ij}, p^{ij}) + \mathcal{H}_s(\phi, \pi_\phi)\right) + \text{bd terms.}$$
The scalar field does not mix with the metric fluctuations at this order.
- Gauge conditions at $\mathcal{O}(\kappa)$: $p + \bar{h}_{ij}\hat{P}^{ij} = 0$ and $\partial_i\left(\hat{g}^{1/d}\bar{h}^{ij}\right) = 0$. *After gauge fixing, the scalar field is gauge invariant and local degree of freedom*.
$$\mathcal{H}_s = \int \mathrm{d}^d x \left[\frac{\hat{N}}{2}\left(\frac{\pi_\phi^2}{\sqrt{\hat{g}}} + \hat{g}^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2\right) + \pi_\phi \hat{N}^i\partial_i\phi\right].$$

Scalar field solution in the exterior regions

- The scalar field expansion in the exterior region I [see e.g., Papadodimas, Raju]

$$\phi(t,r,\varphi) = \sum_{q\in\mathbb{Z}} \int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}} \Big(a_{\omega q} F_{\omega q}(t,r,\varphi) + \text{c.c.} \Big).$$

- $F_{\omega q}(t,r,\varphi)$ are normalizable mode functions with respect to the Klein-Gordon norm and are given in terms of hypergeometric functions

$$\begin{aligned} F_{\omega q}(t,r,\varphi) =& \frac{1}{N_{\omega q}\sqrt{2\pi R_h}} \mathrm{e}^{-\mathrm{i}\omega t - \mathrm{i}q\varphi} (\rho-1)^{-\mathrm{i}\omega/2\eta} \rho^{-\frac{1}{2}(\Delta_+ - \frac{\mathrm{i}\omega}{\eta})} \times \\ & \times {}_2F_1\left(\tfrac{1}{2}(\Delta_+ - \tfrac{\mathrm{i}\omega}{\eta} + \tfrac{\mathrm{i}q\ell^{-1}}{\eta}), \tfrac{1}{2}(\Delta_+ - \tfrac{\mathrm{i}\omega}{\eta} - \tfrac{\mathrm{i}q\ell^{-1}}{\eta}); \Delta_+; \frac{1}{\rho} \right), \\ N_{\omega q} =& \left| \frac{\Gamma(\Delta_+)\,\Gamma(\mathrm{i}\omega/\eta)}{\Gamma\left(\frac{1}{2}(\Delta_+ + \frac{\mathrm{i}\omega}{\eta} - \frac{\mathrm{i}q\ell^{-1}}{\eta})\right) \Gamma\left(\frac{1}{2}(\Delta_+ + \frac{\mathrm{i}\omega}{\eta} + \frac{\mathrm{i}q\ell^{-1}}{\eta})\right)} \right|. \end{aligned}$$

- $\rho = (r/R_h)^2$, $\eta = \ell^{-1}\sqrt{M/2\pi}$ is the surface gravity of the BTZ black hole, and $\Delta_+ = 1 + \sqrt{1+\ell^2 m^2}$.

Scalar field solution in the exterior regions

- The operator $\mathcal{O}$ in the CFT associated to the right boundary that is dual to the scalar field is a generalized free field with dimension Δ_+ .
- By the extrapolate dictionary, we have

$$\begin{aligned}\mathcal{O}(t,\varphi) &= \lim_{r\to\infty} r^{\Delta_+}\phi(t,r,\varphi)\,,\\ &= \frac{R_h^{\Delta_+}}{\sqrt{2\pi R_h}}\sum_{q\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}}\frac{1}{N_{\omega q}}\left(a_{\omega q}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}q\varphi}+\mathrm{c.c.}\right).\end{aligned}$$

- The scalar field expansion in exterior region II and the dual operator $\tilde{\mathcal{O}}$ in the CFT associated to the left boundary is

$$\phi(\tilde{t},\tilde{r},\varphi) = \sum_{q\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}}\Big(\tilde{a}_{\omega q}\tilde{F}_{\omega q}(\tilde{t},\tilde{r},\varphi)+\mathrm{c.c.}\Big),$$

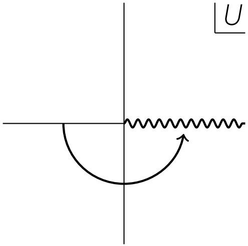
$$\tilde{\mathcal{O}}(\tilde{t},\varphi) = \frac{R_h^{\Delta_+}}{\sqrt{2\pi R_h}}\sum_{q\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}}\frac{1}{N_{\omega q}}\left(\tilde{a}_{\omega q}\mathrm{e}^{-\mathrm{i}\omega\tilde{t}}\mathrm{e}^{-\mathrm{i}q\varphi}+\mathrm{c.c.}\right).$$

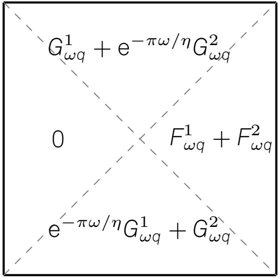
Scalar field solution in the interior

- The mode functions $F, \tilde{F}$ have a logarithmic branch point at $U = 0$ and $V = 0$.

$$\phi(U,V,\varphi) \approx \frac{1}{\sqrt{2\pi R_h}} \sum_{q\in\mathbb{Z}} \int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}}\, a_{\omega q} \mathrm{e}^{-\mathrm{i}q\varphi} \left(\mathrm{e}^{\mathrm{i}\delta_{\omega q}} (2\eta V)^{-\mathrm{i}\omega/\eta} + \mathrm{e}^{-\mathrm{i}\delta_{\omega q}} (-2\eta U)^{\mathrm{i}\omega/\eta}\right) + \text{c.c.}$$

- One can analytically continue the exterior solution a la Unruh to get the scalar field solution in the interior regions F & P.





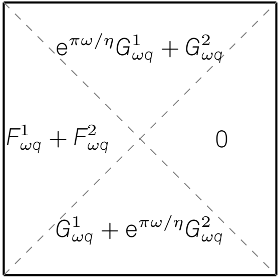


Figure: **Left**: Analytic continuation in the lower-half U-plane. **Center**: The analytic continuation of Region I mode $F_{\omega q} \sim F^1_{\omega q} + F^2_{\omega q}$ to rest of the spacetime. **Right**: The analytic continuation of Region II conjugate modes $\tilde{F}^*_{\omega q} \sim F^1_{\omega q} + F^2_{\omega q}$.

The Hartle-Hawking mode functions

- Upon analytic continuation in lower-half U and V planes, one obtains

$$\phi(U,V,\varphi) = \frac{1}{\sqrt{2\pi R_h}} \sum_{q\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}\omega \left(c_{\omega q} h_{\omega q}(U,V,\varphi) + \mathrm{c.c.} \right).$$

- The Hartle-Hawking modes form a complete set of functions for $-\infty < \omega < \infty$.

$$h_{\omega q}(U,V,\varphi) = \frac{1}{\sqrt{4\pi\omega}\sqrt{2\sinh(\pi\omega/\eta)}} \times \begin{cases} \mathrm{e}^{\pi\omega/2\eta}(F^1_{\omega q} + F^2_{\omega q}) & \mathrm{I} \\ \mathrm{e}^{\pi\omega/2\eta} G^1_{\omega q} + \mathrm{e}^{-\pi\omega/2\eta} G^2_{\omega q} & \mathrm{F} \\ \mathrm{e}^{-\pi\omega/2\eta} G^1_{\omega q} + \mathrm{e}^{\pi\omega/2\eta} G^2_{\omega q} & \mathrm{P} \\ \mathrm{e}^{-\pi\omega/2\eta}(F^1_{\omega q} + F^2_{\omega q}) & \mathrm{II}\,. \end{cases}$$

- $F^i_{\omega q}$ are in terms of hypergeometric functions which are regular near the horizon $r = R_h$ in the exterior region $r > R_h$ and G^i are simiarly regular but in the interior $r < R_h$.

Hartle-Hawking modes

- The Hartle-Hawking state or vacuum is defined as the one annihilated by the $c_{\omega q}$ for all $-\infty < \omega < \infty$:

$$c_{\omega q}|\mathrm{HH}\rangle = 0\,, \quad \text{for} \quad q \in \mathbf{Z}\,, \quad -\infty < \omega < \infty\,.$$

- $c_{\omega q}$ for $\omega < 0$ are sometimes written as $\tilde{c}_{-\omega q}$:

$$\tilde{c}_{\omega q} := c_{-\omega q}\,, \quad \omega > 0\,.$$

- The $c_{\omega q}, \tilde{c}_{\omega q}$ operators for $\omega > 0$ are related to the $a_{\omega q}, \tilde{a}_{\omega q}$ in the original exterior solutions by the Bogoliubov transformations

$$c_{\omega q} = \frac{a_{\omega q} e^{\pi\omega/2\eta} - \tilde{a}^{\dagger}_{\omega q} e^{-\pi\omega/2\eta}}{\sqrt{2\sinh(\pi\omega/\eta)}}\,, \qquad \tilde{c}_{\omega q} = \frac{\tilde{a}_{\omega q} e^{\pi\omega/2\eta} - a^{\dagger}_{\omega q} e^{-\pi\omega/2\eta}}{\sqrt{2\sinh(\pi\omega/\eta)}}\,.$$

- Recall that the Hartle-Hawking state is a highly entangled state in the Hilbert space of the two CFTs as is evident from the above Bogoliubov transformations.

A smooth, global expression for the scalar field

- The Hartle-Hawking modes $h_{\omega q}(U, V, \varphi)$ are not differentiable at the horizon and hence do not solve the KG equation in a neighbourhood of the horizon.
- For instance, in region I, we have the near-horizon behaviour

$$h_{\omega q}(U, V, \varphi) \sim \frac{\mathrm{e}^{-\mathrm{i}q\varphi}\mathrm{e}^{\pi\omega/2\eta}}{\sqrt{4\pi\omega}\sqrt{2\sinh(\pi\omega/\eta)}}\left(\mathrm{e}^{\mathrm{i}\delta_{\omega q}}(2\eta V)^{-\mathrm{i}\omega/\eta} + \mathrm{e}^{-\mathrm{i}\delta_{\omega q}}(-2\eta U)^{\mathrm{i}\omega/\eta}\right).$$

- Clearly, the U and V diverge as $1/U$ and $1/V$ respectively as $UV \to 0$. To cure the divergence, we smear the HH modes with Hermite functions $\psi_n(\omega/\eta)$ in the frequency space

$$g_{nq}(U, V, \varphi) := \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\eta} h_{\omega q}(U, V)\psi_n(\omega/\eta).$$

- The near-horizon behaviour is[1]

$$g_{nq}(U, V, \varphi) \sim \mathrm{e}^{-\mathrm{i}q\varphi}\frac{(-\mathrm{i})^n}{2\pi\sqrt{2\eta}}\int_{-\infty}^{\infty} \mathrm{d}\vartheta \left(\mathrm{e}^{2\mathrm{i}\eta V\mathrm{e}^{-\vartheta}} + \mathrm{e}^{-2\mathrm{i}\eta U\mathrm{e}^{\vartheta}}\right)\psi_n(\vartheta).$$

[1] shown explicitly in the large mass limit

A smooth, global expression for the scalar field

- The scalar field has a smooth global expansion in terms of discrete modes

$$\phi(U,V,\varphi) = \frac{1}{\sqrt{2\pi\ell}} \sum_{q\in\mathbb{Z}} \sum_{n=0}^{\infty} \left(\mathrm{e}_{nq}\, g_{nq}(U,V,\varphi) + \mathrm{c.c.} \right) ,$$
$$\mathrm{e}_{nq} = \frac{1}{\sqrt{\eta}} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{c}_{\omega q} \psi_n(\omega/\eta), \qquad [\mathrm{e}_{mq}, \mathrm{e}^{\dagger}_{nq'}] = \delta_{mn}\delta_{qq'} .$$

- Since the operators e_{nq} are linear combinations of only the $c_{\omega q}$ but not of the $c^{\dagger}_{\omega q}$, they also annihilate the HH vacuum:

$$\mathrm{e}_{nq}|\mathrm{HH}\rangle = 0 , \quad \text{for all} \quad n,\, q .$$

- This scalar field expansion solves the KG equation everywhere in the (U, V) coordinates. Since the $(x, \bar{t})$ coordinates are related to Kruskal coordinates by a smooth diffeomorphism in the region R_0, we have solved the KG equation in the wormhole coordinates.
- The mode expansion is in terms of the discrete mode functions $\hat{g}_{nq}(\bar{t}, x, \varphi) = g_{nq}(U(\bar{t}, x), V(\bar{t}, x), \varphi)$.

The maximal slicing Hamiltonian

- The scalar field solution in wormhole coordinates

$$\phi(\bar{t}, x, \varphi) = \frac{1}{\sqrt{2\pi}\ell} \sum_{q \in \mathbb{Z}} \sum_{n=0}^{\infty} \Big(e_{nq}\, \hat{g}_{nq}(\bar{t}, x, \varphi) + \text{c.c.} \Big).$$

- The scalar field Hamiltonian is given by

$$H = \int \mathrm{d}^d x \left(\frac{N}{2} \left(\frac{\pi_\phi^2}{\sqrt{g}} + g^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2 \right) + \pi_\phi N^i \partial_i \phi \right).$$

with the metric, lapse and shift in the wormhole coordinates.

- In terms of above discrete set of smooth mode functions the scalar field Hamiltonian is a finite, time-dependent Hermitian operator

$$H = \sum_{q,q'} \sum_{n,n'} \begin{pmatrix} e^\dagger_{n'q'} & e_{n'q'} \end{pmatrix} \begin{pmatrix} A_{nn'qq'}(\bar{t}) & B^*_{nn'qq'}(\bar{t}) \\ B_{nn'qq'}(\bar{t}) & A^*_{nn'qq'}(\bar{t}) \end{pmatrix} \begin{pmatrix} e_{nq} \\ e^\dagger_{nq} \end{pmatrix}.$$

- The kernels A and B are finite (expressions on the next slide).

The maximal slicing Hamiltonian

- The kernels A and B are given by

$$A_{nn'qq'}(\bar{t}) = \frac{1}{2\pi\ell^2}\int \mathrm{d}x\mathrm{d}\varphi\sqrt{g}\Bigg[N\Big(\mathcal{D}_{\bar{t}}\hat{g}^{*}_{n'q'}\mathcal{D}_{\bar{t}}\hat{g}_{nq} + g^{xx}\hat{g}'^{*}_{n'q'}\hat{g}'_{nq} + \left(m^2 + qq'g^{\varphi\varphi}\right)\hat{g}^{*}_{n'q'}\hat{g}_{nq}\Big) + 2N^{x}\mathcal{D}_{\bar{t}}\hat{g}^{*}_{n'q'}\hat{g}'_{nq}\Bigg],$$

$$B_{nn'qq'}(\bar{t}) = \frac{1}{2\pi\ell^2}\int \mathrm{d}x\mathrm{d}\varphi\sqrt{g}\Bigg[N\Big(\mathcal{D}_{\bar{t}}\hat{g}_{n'q'}\mathcal{D}_{\bar{t}}\hat{g}_{nq} + g^{xx}\hat{g}'_{n'q'}\hat{g}'_{nq} + \left(m^2 + qq'g^{\varphi\varphi}\right)\hat{g}_{n'q'}\hat{g}_{nq}\Big) + 2N^{x}\mathcal{D}_{\bar{t}}\hat{g}_{n'q'}\hat{g}'_{nq}\Bigg].$$

where $\mathcal{D}_{\bar{t}} = N^{-1}(\partial_{\bar{t}} - N^{x}\partial_{x})$ is the normal derivative and $' = \partial_x$.

The maximal slicing Hamiltonian

- The Hamiltonian $H(\bar{t})$ is time dependent and explicitly Hermitian.
- The time evolution operator is unitary and is given by

$$U(\bar{t}) = \bar{\mathcal{T}} \exp\left(-\mathrm{i} \int_0^{\bar{t}} \mathrm{d}\bar{u}\, H(\bar{u})\right).$$

- The symbol $\bar{\mathcal{T}}$ stands for anti-time ordering which is appropriate here since H is a Heisenberg Hamiltonian.
- In the Heisenberg picture, the wave function at all times is the Hartle-Hawking state $|\mathrm{HH}\rangle$ and the operators evolve unitarily with this Hamiltonian, e.g., $e_{nq}(\bar{t}) = U(\bar{t})^\dagger e_{nq}(0) U(\bar{t})$.
- In the Schroedinger picture, the state $|\mathrm{HH}; \bar{t}\rangle = U(\bar{t})|\mathrm{HH}\rangle$ remains pure at all times.
- This Hamiltonian describes the unitary time development of operators / states as they traverse the horizon.

Wightman functions in wormhole coordinates

- Using the scalar field expansion in terms of discrete smooth mode functions $\hat{g}(\bar{t}, x, \varphi)$, the Wightman function is given by

$$\langle \mathrm{HH}|\phi(\bar{t}, x, \varphi)\phi(\bar{t}', x', \varphi')|\mathrm{HH}\rangle = \frac{1}{2\pi\ell^2}\sum_{n,q} \hat{g}_{nq}(\bar{t}, x, \varphi)\, \hat{g}^*_{nq}(\bar{t}', x', \varphi').$$

- An alternate expression is obtained by using the method of images (the BTZ geometry is a quotient of AdS_3 by a discrete group of translations) [Ichinose-Satoh]

$$\begin{aligned} G(\bar{t}, x, \varphi; \bar{t}', x', \varphi') =& \langle \mathrm{HH}|\phi(\bar{t}, x, \varphi)\phi(\bar{t}', x', \varphi')|\mathrm{HH}\rangle \\ =& \sum_{n\in\mathbf{Z}} \frac{1}{\sqrt{z_n^2 - 1}} \left(z_n + \sqrt{z_n^2 - 1}\right)^{1-\Delta_+}, \end{aligned}$$

$$\begin{aligned} z_n =& \frac{1}{(1+\eta^2 UV)(1+\eta^2 U'V')} \times \\ & \left((1-\eta^2 UV)(1-\eta^2 U'V') \cosh\left(\ell\eta(\varphi - \varphi' + 2n\pi)\right) + 2\eta^2(VU' + UV')\right). \end{aligned}$$

Wightman functions in wormhole coordinates

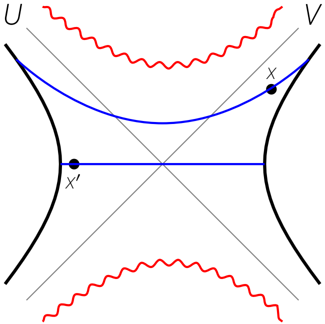
- The two expressions are equivalent upon coordinate transformation from Kruskal-Szekeres to wormhole coordinates, i.e., $U = U(\bar{t}, x)$, $V = V(\bar{t}, x)$, $U' = U(\bar{t}', x')$, $V' = V(\bar{t}', x')$ with

$$U(\bar{t}, x) = \pm\eta^{-1} \exp\left[-\eta\bar{t} - R_h \int_x^\infty \frac{\mathrm{d}y}{(y^2 - R_-^2)\sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+^2 - R_-^2}} + y\right)\right],$$

$$V(\bar{t}, x) = \pm\eta^{-1} \exp\left[\eta\bar{t} + R_h \int_x^\infty \frac{\mathrm{d}y}{(y^2 - R_-^2)\sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+^2 - R_-^2}} - y\right)\right].$$

2-point function: 1st example

$$G(\bar{t}, x_0, 0; 0, -x_0, 0) = \langle \mathrm{HH}| \phi(\bar{t}, x = x_0, \varphi = 0)\, \phi(\bar{t}' = 0, x' = -x_0, \varphi' = 0) |\mathrm{HH}\rangle.$$



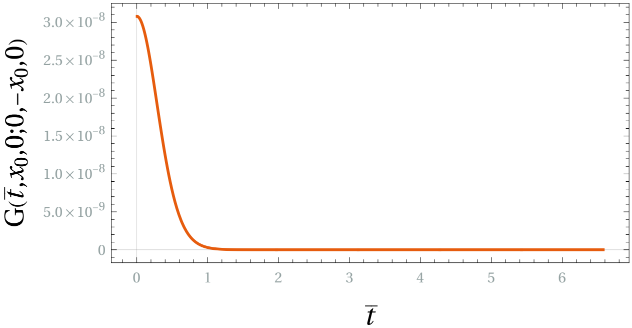


Figure: The correlation function plotted as a function of the time $\bar{t}$, keeping $x_0 = 10R_h$ fixed. The real part (orange) decays exponentially. The imaginary part simply vanishes.

2-point function: 2nd example

$$G(\bar{t},0,0;0,0,0)=\langle \mathrm{HH}|\phi(\bar{t},x=0,\varphi=0)\,\phi(\bar{t}'=0,x'=0,\varphi'=0)|\mathrm{HH}\rangle.$$

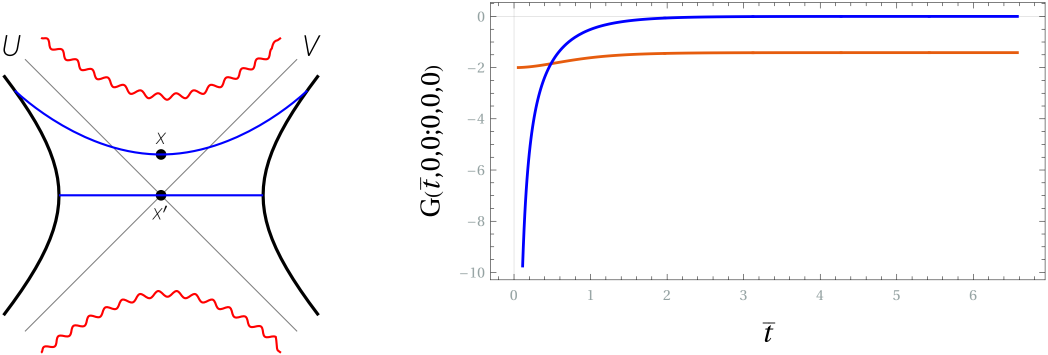


Figure: The auto-correlation function plotted as a function of time $\bar{t}$. The imaginary part (blue) has the UV divergence at small times but decays at large times. On the other hand the real part (orange) saturates to a finite non-zero value at late times.

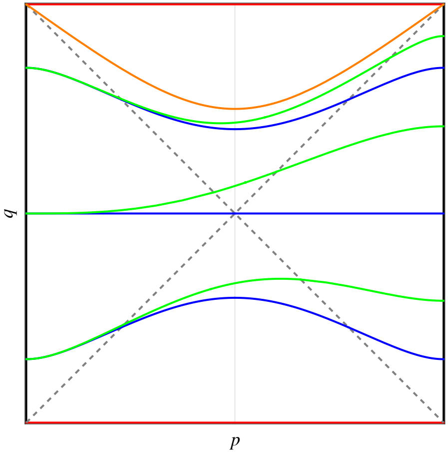
Summary

- We quantize a probe scalar field in the maximally sliced BTZ black hole in terms of a complete set of discrete mode functions which are smooth across the horizons. This was done by smearing the singular Hartle-Hawking mode functions in the frequency domain with Hermite functions.
- We write down the scalar field Hamiltonian in the Heisenberg picture which propagates the field along maximal slices in terms of the discrete set of smooth mode functions. This Hamiltonian is a finite, Hermitian and gauge-invariant operator in the product of the two CFTs associated to the two boundaries and describes unitary time evolution of CFT states / operators.
- We calculate Wightman functions in the Hartle-Hawking state. Not all correlation functions decay in the maximal slicing time $\bar{t}$.

Thank you. Questions?

Large Diffeomorphisms

- Time translation on the right boundary uniquely continues into the bulk.



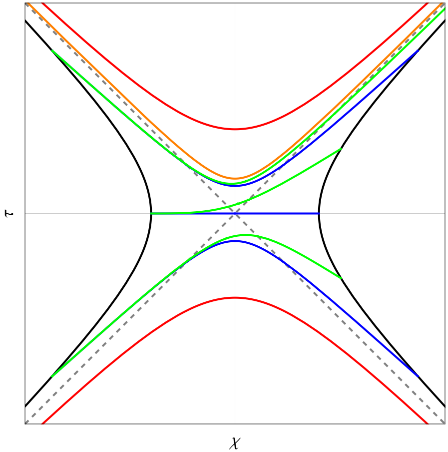


Figure: The action of the large diffeomorphism corresponding to time translations only on the right boundary. The blue curves are the original slices and the green curves are their unique images under right time translation (small diffeos are completely fixed).

Large Diffeomorphisms: States and operators

- Under the large diffeomorphism, the states and operators evolve by the following evolution operator (with $c = 1, \tilde{c} = 0$)

$$U = \bar{\mathcal{T}} \mathrm{e}^{-\mathrm{i}\int_0^u \mathrm{d}\bar{u} H(\bar{u})},\; H = \int \mathrm{d}^2 x \left[\zeta_\perp \left(\frac{\pi_\phi^2}{2\sqrt{g}} + \tfrac{1}{2}\sqrt{g}(g^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2) \right) + \zeta^i \pi_\phi \partial_i \phi \right],$$

$$\zeta_\perp(u,x) = \begin{cases} \frac{x\sqrt{x^2+R_+^2-R_-^2}}{\ell\sqrt{x^2+R_+^2}} \left[c + \ell^3 \partial_u T_{c,\tilde{c}} \int_x^\infty \frac{\mathrm{d}y}{y^2} \frac{\sqrt{y^2+R_+^2}}{(y^2+R_+^2-R_-^2)^{3/2}} \right] & x > 0 \\ -\frac{x\sqrt{x^2+R_+^2-R_-^2}}{\ell\sqrt{x^2+R_+^2}} \left[\tilde{c} + \ell^3 \partial_u T_{c,\tilde{c}} \int_{-x}^\infty \frac{\mathrm{d}y}{y^2} \frac{\sqrt{y^2+R_+^2}}{(y^2+R_+^2-R_-^2)^{3/2}} \right] & x < 0 \end{cases},$$

$$\zeta^x(u,x) = \frac{1}{x}\left(\zeta_\perp(u,x) T_{c,\tilde{c}}(u) + R_+ \partial_u R_+\right), \qquad \zeta^\varphi(u,x) = 0,$$

with the function $T_{c,\tilde{c}}(u)$ defined implicitly by

$$(c+\tilde{c})u = -2T_{c,\tilde{c}} \int_{R_+(T_{c,\tilde{c}})}^\infty \frac{\mathrm{d}\rho}{\rho} \left(\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} \right)^{-1} \left(\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} + \frac{T_{c,\tilde{c}}^2}{\rho^2} \right)^{-1/2}.$$