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Canonical presentation of homogeneous cosmological space-times, A review note

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Abstract

A review of the canonical formulation of GR was envisaged in the discussion meeting at CMI during Aug 25-29. Alok gave an introduction in a general context. This was illustrated in the context of homogeneous cosmologies. These provide good, tractable examples to discuss both the classical aspects of systems with a Hamiltonian constraint as well as issues that arise in the quantization process. These are summarized with quantization based on Wheeler-de Witt equation.

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1. INTRODUCTORY REMARKS:

Why quantize gravity? No direct motivation from experiments – no black body radiation, no interference fringes, no bearing on stability of atoms, all known sources and detectors are macroscopic with large masses (compare with photons).

However, the successful GR predicts its inadequacy in two extreme situations: complete gravitational collapse and early universe. These regimes are inherently non-perturbative i.e cannot be analyzed as quantum corrections to a given background space-time. Yet another reason for quantum gravity may be advanced. Hawking effect, without quantum gravity, suggests that a pure collapsing state forming a black hole may evolve into a mixed state after complete evaporation. If so, this contradicts expectation based on general quantum framework. A possible resolution maybe obtained with the help of a quantum theory of gravity.

So while, one *can* address how to include *quantum gravity corrections* to backgrounds, their experimental verification is exceedingly hard. To some extent, such work has been done by Donoghue [1], but verification is beyond experimental capability. These corrections to gravitational potential are of order $(\ell_P/r)^2$.

The non-renormalizability is less of a bar for estimating these corrections in a effective field theory framework. However, for any estimates in the *non-perturbative regimes*, we cannot restrict to an effective scale and invoke EFT.

Why Canonical methods? Unlike the path integral quantization which is directly adaptable to semiclassical corrections, the canonical methods do not presume a background in its formulation. So amenable to a non-perturbative formulation.

The framework is well adapted to isolating symmetries, gauges, physical degrees of freedoms.

However, it is not naturally adapted to possible "topology changing processes". Path integral methods are amenable to these processes but usually only with the help of Euclidean continuation.

Canonical view: Alok has already discussed the basic ADM formulation which uses the Einstein-Hilbert action with 3+1 decomposition. This leads to canonical form of the theory with first class constraints. Here is a brief recap:

Brief Digression on Hyperbolicity (See Wald [2]):

The analysis of possible causal pathologies such as lack of time orientation ("chronology condition"), presence of closed causal curves ("causality condition"), presence of almost closed causal curves ("strong causality condition"), presence of closed non-causal curves which can become closed due to a widening of the light cones in a neighboring metric ("stable causality"). Stable causality implies that the topology of the space-time must be $\mathbb{R} \times \Sigma$. The demand that all events in the space-time should have their causes recorded on a space-like hypersurface and *only* on that hypersurface implies space-time admits a Cauchy surface. This is synonymous with "global hyperbolicity". The last condition implies that such space-times are arenas for the non-gravitational physics to provide unambiguous causality and predictability. The well-posedness of the initial value problem for Einstein equations implies that local evolution constructs a causally well behaved solution. This analysis also identifies $G_{\mu 0} = 0$ as the "non-dynamical" equations (will connect to primary constraints) and the remaining six equations as the true, second order evolution equations for the spatial metric. That Einstein (differential) equations can also be cast in a Hamiltonian form is an additional feature of Einstein equations. The initial value problem identifies the 3-metric and its time derivative as initial data subject to the $G_{\mu 0} = 0$ equation. This can always be done locally in both space and time and extended to larger domains to lead to "maximal Cauchy development". This is guaranteed to be a globally hyperbolic space-time. If such space-times are extended (real) analytically, the globally hyperbolic space-time has Cauchy horizons as its boundary. Admissibility of a Canonical formulation is a non-automatic property of any field theory. Even if the equations are derivable from an action principle without non-cancellable surface terms, there may be constraints and the constraint analysis may fail to be consistent (Dirac Yeshiva lectures). In some cases, symplectic structure may fail to exist. Einstein Hilbert action with Gibbons-Hawking surface term, turns out to be a constrained theory with a closed Poisson bracket algebra of constraints.

3+1 decomposition and summary:

On the backdrop of these, canonical formulation begins with manifold which is foliated by 3-surfaces, labeled by some function $T : M \to \mathbb{R}$. The ADM parametrization of the metric, $ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$, with g_{ij} positive definite, guarantees that the t =constant hypersurfaces, Σ_t are space-like. Then the following definitions and equations follow.

$$\begin{aligned}
S[g_{ij}, N, N^i] &= \int dt \int_{\Sigma_t} d^3x \ N\sqrt{g}\left\{R[g] - k^{ij}k_{ij} + (k^{ij}g_{ij})^2\right\} \ ; && (1.1)\\
k_{ij} &= \frac{2}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \ = \ \pi^{ij} = \sqrt{g}N\left(k^{ij} - (k^{mn}g_{mn})g^{ij}\right) ; && (1.2)\\
H[N] &:= \int_{\Sigma_t} N\left\{-\sqrt{g}R[g] + \frac{\pi^{ij}\pi_{ij} - \frac{\pi^2}{2}}{\sqrt{g}}\right\} \ , \text{ (Hamiltonian constraint)}, && (1.3)\\
H[\vec{N}] &:= \int_{\Sigma_t} d^3x N_i\left\{-2\nabla_j \pi^{ij}\right\} \ , \ \text{(diffeomorphism constraint)} && (1.4)\\
H_{can} &= H[N] + H[\vec{N}] && (1.5)\\
\{H[\vec{M}], H[\vec{N}]\} &= -H[\vec{L}] \ , \ \vec{L} := \vec{M}\cdot\vec{\nabla}\vec{N} - \vec{N}\cdot\vec{\nabla}\vec{M} \ ; && (1.6)\\
\{H[\vec{M}], H[N]\} &= -H[\vec{K}] \ , \ \vec{K} := \vec{M}\cdot\vec{\nabla}N; && (1.7)\\
\{H[M], H[N]\} &= H[\vec{L}] \ , \ L^i := g^{ij}(N\partial_j M - M\partial_j N). && (1.8)
\end{aligned}$$

These have been discussed by Alok in detail. While Alok introduced the canonical formulation in terms of embedding a 3-surface Σ in a space-time with some metric such that the embedding was space-like with respect to that metric, for presentation of canonical formulation, this is not necessary.

Alok also emphasized that (a) the constrain algebra is *not* a Lie algebra because the last bracket involves $\vec{L}$ which involves the metric i.e the parameter of the constraint is dependent on the phase space variable instead of being constants in the phase space. The closure of the Poisson bracket algebra implies that while the constraints do constitute a first class system, they do not form a Lie algebra. The 1+1 GR without matter however is an exception and the constraint algebra is a Lie algebra. Alok also explained how the vector constraints generate the action of infinitesimal diffeomorphisms on the phase space tensors, g_{ij}, π^{ij} (density weight 1). For the interpretation of the Hamiltonian constraint as generator of evolution, the equations of motion must be used.

Remark: For a space-time with say asymptotic boundaries, certain fall off behaviors have to be specified either for mathematical reasons or for physical reasons eg asymptotically flat/AdS/dS space-times. These conditions themselves allow transformations among the (conformal) boundary data which are then considered as *asymptotic symmetries.* Vector fields generating infinitesimal diffeomorphisms may vanish asymptotically or have a non-zero limit inducing a symmetry transformation. The former ones are termed *gauge generators* as they leave any particular boundary data unchanged. Thus, the lapse and shift vectors that vanish at the asymptotic boundary generate the same space-time. Non-vanishing shift vector ones generates a symmetry transformation on the spatial boundary data while non-vanishing lapse generates "true evolution". For Σ compact without boundaries, such "true generators" do not exist.

Although the above summary does not include matter and cosmological constant, the canonical formalism extends to these as well.

To get some feeling we discuss a few toy models with finitely many degrees of freedom, "mini-superspaces", and also discuss the issues that arise in their "Wheeler-de Witt quantization".

2. COSMOLOGICAL MINI-SUPERSPACES

Both as an illustration of the canonical form of GR and as an exploration of its quantization, we specialize to the class of spatially homogeneous spacetimes, not necessarily solutions of

Einstein equations. In its canonical presentation, these constitute *mini-superspace models.* We specify the class of such space-times; reduce of the full action to the minisuperspace action and present its canonical form. In considering their solutions, we will meet the first feature of "problem of time". This is followed by their Quantization and issues involved. This will bring out other features of the problem of time.

Spatially homogeneous spacetimes:

These are spacetimes that have isometries whose orbits are space-like hypersurfaces. Equivalently, such spacetimes are foliated by spatial slices on each of these there is a transitive action of an isometry group group G i.e. from any chosen point, any other point (possibly itself) on a hypersurface can be reached by a transformation by at least group element. In general there are many group elements effecting such a transformation. If it so happens that G or a subgroup $G^* \subset G$ action is one-to-one, the group action is said to be *simply transitive* and the space-time is said to be of *Bianchi* type. If neither G nor any of its subgroups has simply transitive action, the space-time with such an isometry group is said to be of *Kantowski-Sachs* type. The Kantowski-Sachs type space-time arises only in the special case where the hypersurface has topology of $S^2 \times \mathbb{R}$ and $G = SO(3) \times \mathbb{R}$. A (vacuum) space-time solution of this class is the interior of the Schwarzschild space-time. Transitive action already implies that the dimension of any isometry group must be ≥ 3. Maximum number of Killing vectors in 3 dimensions cannot exceed 6. Simple transitivity implies that the isometry groups must be exactly 3 dimensional. Thus, Bianchi groups are 3-dimensional while Kantowski-Sachs spacetimes have higher dimensional isometry group. All 3-dimensional groups have been classified by Bianchi ($\sim$ 1897) into 9 classes. Each of the spatial slices are identified with the group manifolds themselves.

A. Classification of the Bianchi Lie Algebras

A group manifold comes equipped with left (and right) invariant vector fields, X_I and their dual invariant 1-forms $E^I, E^I(X_J) = \delta^I_J$. The E^I's are called Maurer-Cartan forms and for matrix groups, these can be simply computed by from $g^{-1}dg$. They satisfy the following relation, defining the structure constants C^K_{IJ}.

$$[X_I, X_J] \ = \ C^K_{\ IJ} X_K \ \leftrightarrow \ dE^I \ = \ -\frac{1}{2} C^I_{\ JK} E^J \wedge E^K, \ \text{(Maurer-Cartan)} \tag{2.1}$$

$$\text{where,} \qquad C^K_{\ IJ} = -C^K_{\ JI} \ ; \ \sum_{(IJK)} C^M_{\ IL} C^L_{JK} \ = \ 0 \ \text{(Jacobi).} \tag{2.2}$$

The equivalence in the first equation follows using the definition, $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

The classification of the Bianchi algebras proceeds by writing $C^K_{IJ} = C^{KL}\mathcal{E}_{LIJ}$ with $C^{IJ} := M^{IJ} + \mathcal{E}^{IJK} A_K, \ M^{IJ} = M^{JI}$. The Jacobi identity implies that $M^{IJ} A_J = 0$. The $A_I = 0$ defines the class A Bianchi groups and non-zero A_I defines the class B Bianchi groups. .

The indices transform under $GL(3)$. Thus M^{IJ} transforms as a symmetric tensor of rank 2. By a general linear (tensor) transformation, M^{IJ} can be diagonalised. Further doing scaling, the non-zero diagonal elements can be brought to ± 1. It is conventional to write $M^{IJ} = n^I \delta^{IJ}, n^I = 0, \pm 1$. These are in a specific basis in the Lie algebra (modulo permutations) and no further tensor transformations are envisaged. Some of these integers can be zero indicating the rank of M^{IJ}. Using these, we conventionally write $C^I_{\ JK} := \mathcal{E}_{JKI}(n^I + a\delta^I_1)$.

For the class A, there are six possibilities, modulo permutations, organized by the possible ranks of M, (0,1,2,3) and two possible signatures when the ranks are 2 and 3. For class B, the possible ranks are (0,1,2) and two possible signatures for rank 2. The two most commonly studied cases are the type I model which has zero rank and the type IX model which has rank 3.

For type I, all isometries commute. Globally, the group manifolds can be $\mathbb{R}^3$ of T^3. The *Kasner solution* belongs to this class. With the $\mathbb{R}^3$ topology, there is the further isometry stability group of $SO(3)$ which has the spatially flat FRW space-time.

Type IX, also admits an isotropy subgroup of $SO(3)$ and contains the closed, FRW space-time. The Open, FRW space-time belongs to class B, type V.

Here is a summary table (Landau-Lifshitz [3]).

Type	a	n_1	n_2	n_3	Remarks
Class A					
I	0	0	0	0	Abelian, Euclidean space/T^3
					(Leads to Kasner space-time)
II	0	1	0	0	Nilpotent (Heisenberg)
VII_0	0	1	1	0	
VI_0	0	1	-1	0	
IX	0	1	1	1	semi-simple compact, S^3 is special
					with isotropy, Central to BKL
VIII	0	1	1	-1	Semi-simple, non-compact ($sl(2,\mathbb{R})$)
Class B					
V	1	0	0	0	H^3 a special case (with isotropy)
IV	1	0	0	1	
VII_a	a	0	1	1	
III	1	0	1	-1	sub-case of type VI_a
VI_a	a	0	1	-1	

It turns out that for class B space-times, the action principle is ill-defined due to the clash of homogeneity and non-vanishing surface term, see [4, 5]. From now on, we restrict to class A spacetimes only.

B. Left Invariant Riemannian geometry on the Bianchi group manifold:

Let γ_{IJ} be an arbitrary, invariant (constant), symmetric positive definite metric tensor on the group manifold and let ∇ be a covariant derivative with a connection $\Gamma^I{}_{JK}$, defined through, $\boxed{\nabla_{X_J} X_K := \Gamma^I{}_{JK} X_I}$ which is torsion free and compatible with γ_{IJ}:

$$T^I{}_{JK} := \Gamma^I{}_{JK} - \Gamma^I{}_{KJ} - C^I{}_{JK} = \ 0 \ = X_K(\gamma_{IJ}) - \Gamma^L{}_{KI}\gamma_{LJ} - \Gamma^L{}_{KJ} g_{IL} \ .$$

The invariance condition implies that $X_I(\gamma_{JK}) = 0 = X_I(\Gamma^L{}_{JK})$. It follows that,

$$\Gamma^I{}_{JK} = -\frac{1}{2}\gamma^{IL}\left[C_{JKL} + C_{KJL} - C_{LJK}\right] \quad , \quad C_{IJK} := \gamma_{IL}C^L{}_{JK} = \gamma_{IL}\mathcal{E}_{JKL}n^L \ ; \tag{2.3}$$

$$= \frac{1}{2}\mathcal{E}^I{}_{JK}(n^I - n^J + n^K) \ \Rightarrow \ \Gamma^I{}_{IK} = 0. \quad \text{Here,} \quad \mathcal{E}^I{}_{JK} := \gamma^{IL}\mathcal{E}_{LJK} \tag{2.4}$$

$$R^I{}_{JKL} = \Gamma^I{}_{KM}\Gamma^M{}_{LJ} - \Gamma^I{}_{LM}\Gamma^M{}_{KJ} - \Gamma^I{}_{MJ}C^M{}_{KL} \tag{2.5}$$

$$R_{JL} = 0 - \Gamma^I{}_{LM}\Gamma^M{}_{IJ} - \Gamma^I{}_{MJ}C^M{}_{IL} \quad , \quad R = \gamma^{JL}R_{JL} \ . \tag{2.6}$$

On each of the homogeneous hypersurface, Σ_t, we allow the γ_{IJ} metric to vary with t. This also varies the corresponding Riemannian geometry.

C. Most general Homogeneous ansatz and reduction

We are ready to write the most general space-time metric form which has one of the Bianchi groups as its symmetry group and proceed to the canonical formulation. Let (t, x^i) denote the coordinates as usual with the lapse $N(t, \vec{x})$, shift vector $N^i(t, \vec{x})$ and the metric $g_{ij}(t, \vec{x})$. We take the spatial homogeneity ansatz as,

$$g_{ij}(t, \vec{x}) := \gamma_{IJ}(t)E^I_i(\vec{x})E^J_j(\vec{x}) \ , \ N^i(t, \vec{x}) = N^I(t)X^i_I(\vec{x}) \ , \ N(t, \vec{x}) := N(t) \ .$$

This implies that $\sqrt{g}(t, \vec{x}) = \sqrt{\gamma(t)}det(E^I_i(\vec{x}))$ and also $R^i{}_{jkl}(g) = X^i_I E^J_j E^K_k E^L_l R^I{}_{JKL}(\gamma)$. Next, the extrinsic curvature $k_{ij}(t, \vec{x}) = \frac{1}{2N(t)}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) = \frac{1}{2N}\left(\dot{g}_{ij} - (\mathcal{L}_{\vec{N}}g)_{ij}\right)$. Using $\mathcal{L}_{X_J}E^I = i_{X_J}dE^I = C^I{}_{JL}E^L$ and $g = \gamma_{IJ}E^I \otimes E^J$, we get $(\mathcal{L}_{N^J X_J}g)_{ij} = \left[2N^K C^L{}_{K(I}\gamma_{J)L}\right]E^I_i E^J_j$. Hence, $(\mathcal{L}_{\vec{N}}g)_{ij} = (\mathcal{L}_{\vec{N}}g)_{IJ}E^I_i E^J_j$. And $\dot{g}_{ij} = \dot{\gamma}_{IJ}E^I_i E^J_j$ and,

$$k_{ij}(t, \vec{x}) := K_{IJ}(t)E^I_i(\vec{x})E^J_j(\vec{x}) \text{ with } K_{IJ}(t) = \frac{1}{2N(t)}\left(\dot{\gamma}_{IJ}(t) - 2N^K C^L{}_{K(I}\gamma_{J)L}\right) \ . \tag{2.7}$$

It follows that $k_{ij}k^{ij} = K_{IJ}K^{IJ}$, $K = K_{IJ}\gamma^{IJ}$ are functions of t alone. Denote, $V_0 :=$

$\int_\Sigma d^3x\ det(E^I_i)(\vec{x})$. Putting all these together, it follows that

$$
\begin{aligned}
ds^2 &:= -N^2(t)dt^2 + g_{ij}(t)(dx^i + N^i(t)dt)(dx^j + N^j(t)dt)\ ; && (2.8)\\
&= -N^2(t)dt^2 + \gamma_{IJ}(t)\big(E^I(\vec{x}) + N^I(t)dt\big)\big(E^J(\vec{x}) + N^J(t)dt\big). && (2.9)\\
S[g, N, N^i] &= \int dt\ N(t)\int_\Sigma d^3\sqrt{g}\big(R(g) - k^2 + k_{ij}k^{ij}\big) && (2.10)\\
&= \int dt\ N(t)\big[\sqrt{\gamma}\big(R(\gamma) - K^2 + K_{IJ}K^{IJ}\big)\big]V_0\ ; && (2.11)\\
\pi^{ij}(t,\vec{x}) &= \sqrt{g}\big(k^{ij} - kg^{ij}\big)\ =\ det(E^I_i)\underbrace{\left[\sqrt{det\gamma}\big(K^{IJ} - K\gamma^{IJ}\big)\right]}_{P^{IJ}}X^i_I X^j_J\ . && (2.12)\\
&\Rightarrow \frac{1}{V_0}\int_\Sigma d^3x\ \pi^{ij}E^I_i E^J_i\ =:\ P^{IJ}(t) = \sqrt{\gamma}\big(K^{IJ} - K\gamma^{IJ}\big)\ . && (2.13)
\end{aligned}
$$

The t and $\vec{x}$ dependences separate out cleanly. The constant V_0 introduced above could be the invariant volume of the spatial slice if the slice is compact or it could be the volume of some fiducial spatial cell. *At this stage, the simple transitivity etc may be forgotten. Its purpose was to get the ansatz which defined the restricted class of spacetimes.*

The relation between γ, g and between P, π are inverted as,

$$\gamma_{IJ}(t)\ =\ \frac{1}{V_0}\int_\Sigma d^3x\ det(E)g_{ij}(\vec{x})X^i_I X^i_J\ \ ,\ \ P^{IJ}(t)\ =\ \frac{1}{V_0}\int_\Sigma d^3y\ \pi^{kl}(\vec{y})E^K_k E^L_l\ .$$

It is easy to see that the Poisson brackets among the $g_{ij}(\vec{x})$ and $\pi^{kl}(\vec{y})$ imply

$$\{\gamma_{IJ}, P^{KL}\}\ =\ \frac{1}{V_0}\delta^K_{(I}\delta^L_{J)}\ ,\ \{\gamma,\gamma\} = 0 = \{P, P\}\ .$$

Reduction of the Diffeomorphism Constraint

We have,

$$
\begin{aligned}
H[\vec{N}] &= \int_\Sigma d^3x\, N_i(t,\vec{x})\big(-2\nabla_j\pi^{ij}\big)\ =\ -2\int_\Sigma d^3x N_I E^I_i\ E^J_j\nabla_J\big(det(E)P^{KL}X^i_K X^j_L\big) && (2.14)\\
&= N_I(-2\nabla_J P^{IJ})V_0 && (2.15)
\end{aligned}
$$

Referred to the invariant basis,

$$
\begin{aligned}
\nabla_{X_J}P^{IJ} &= X_J(P^{IJ}) + \Gamma^I{}_{JK}P^{KJ} + \Gamma^J{}_{JK}P^{IK} - 1\cdot\Gamma^L{}_{LJ}P^{IJ}\ = \Gamma^I{}_{JK}P^{KJ} && (2.16)\\
&= -\gamma^{IL}P^{JK}\gamma_{JJ'}C^{J'}{}_{KL}\ =\ -\gamma^{IL}P^K_{J'}\mathcal{E}_{KLN}M^{NJ'}\ =\ \gamma^{IL}\mathcal{E}_{KLN}\delta^{NJ'}n^{J'}P^K_{J'}. && (2.17)\\
\therefore H[\vec{N}] &= V_0 N^I\big\{-2C^J{}_{IK}P^K_J\big\} && (2.18)
\end{aligned}
$$

The fourth term is there because the momentum P^{IJ} is of density weight 1 and thus the last two terms cancel out. They are also individually zero thanks to Bianchi A. The first term vanishes because of invariant basis components[1]

If we now specialize to the case where the momentum is diagonal, $P^K_{J'} = P_{J'}\delta^K_{J'}$, then $\delta^{NJ'}\delta^K_{J'}n_{J'}\pi_{J'}$ is symmetric in N, K and the diffeomorphism constraint vanishes. This happens when the metric is diagonal and these models are called the *diagonal Bianchi models*. We restrict to these models from now on.

Reduction of the Hamiltonian Constraint:

The Hamiltonian constraint likewise reduces to,

$$H[N] = N\left(-\sqrt{\gamma}R + \frac{P^{IJ}P_{IJ} - P^2/2}{\sqrt{\gamma}}\right)V_0 \quad , \quad P := P^{IJ}\gamma_{IJ} \ . \tag{2.19}$$

We have thus completed the direct reduction of the full model action and the constraints by using the homogeneous ansatz.

Note: We have another route available from the reduced action (2.12). We can take the reduced action and carry out its Hamiltonian form. Thus we take

$$L(N, \vec{N}, \gamma) := N(t)V_0\sqrt{\gamma}\left[R(\gamma) - K^2 + K^{IJ}K_{IJ}\right] \ , \ K_{IJ} = \frac{1}{2N(t)}\left(\dot{\gamma}_{IJ} - 2N^K C^L_{K(I}\gamma_{J)}L\right) \ . \tag{2.20}$$

Defining the conjugate momentum, $\Pi^{IJ} = \frac{\partial L}{\partial \dot{\gamma}_{IJ}}$, we see that this gives $\boxed{\Pi^{IJ} = V_0 P^{IJ}}$. With this done, the Hamiltonian constraint takes the form,

$$H[N] = N\left(-V_0\sqrt{\gamma}R + \frac{\Pi^{IJ}\Pi_{IJ} - \Pi^2/2}{V_0\sqrt{\gamma}}\right) \quad , \quad \Pi := \Pi^{IJ}\gamma_{IJ} \ \text{ and } \ \{\gamma, \Pi\} = \delta \ . \tag{2.21}$$

Simplification: For class A, diagonal metric, the diffeo constraint vanishes identically and we might as well take the shift to be zero to simplify the expressions eg $K_{IJ} = \frac{1}{2N}\dot{\gamma}_{IJ}, K = \frac{1}{2}\gamma^{IJ}\dot{\gamma}_{IJ}$. This leads to,

$$K^{IJ}K_{IJ} - K^2 = \frac{1}{4N^2}\left(\gamma^{IK}\gamma^{JL}\dot{\gamma}_{IJ}\dot{\gamma}_{KL} - (\gamma^{IJ}\dot{\gamma}_{IJ})^2\right) := \frac{1}{4N^2}G^{IJKL}\dot{\gamma}_{IJ}\dot{\gamma}_{KL} \ . \tag{2.22}$$

$$L(N, \gamma, \dot{\gamma}) = \frac{V_0\sqrt{\gamma}}{4N}G^{IJKL}\dot{\gamma}_{IJ}\dot{\gamma}_{KL} - V_0\sqrt{\gamma}NR(\gamma) \ . \tag{2.23}$$

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[1] General expression for a tensor density of weight w is: $\nabla_j T^{i_1\ldots}{}_{j_i\ldots} = \text{(usual terms)} - w\Gamma^k{}_{kj}T^{i_1\ldots}{}_{j_1\ldots}$.

The G^{IJKL} is called the *supermetric* and it always has *Lorentzian signature*[2]. It is a metric on the space of $\{\gamma_{IJ}\}$ of the diagonal Bianchi class models. More generally, the space of metrics is called the *superspace* and the metric induced on it by the Einstein-Hilbert action is called the *supermetric*. What we have is the special case of a *mini-superspace* and the corresponding supermetric also called the Wheeler-de Witt metric.

D. The Diagonal γ, Misner parametrization

Let the metric $\gamma_{IJ}(t) := diag(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3})$. This gives $K^{IJ}K_{IJ} - K^2 = \frac{1}{N^2}\big(\sum_i \dot{\beta}_i^2 - (\sum_i \beta_i)^2\big)$. This can be diagonalised by introducing the Misner parameters $(\alpha, \beta_\pm)$.

$$\beta_1 = \alpha + \beta_+ + \sqrt{3}\beta_- \ , \ \beta_2 = \alpha + \beta_+ - \sqrt{3}\beta_- \ , \ \beta_3 = \alpha - 2\beta_+ \ ; \tag{2.24}$$

$$\Rightarrow \alpha = \sum_i \beta_i/3 \ , \ \sqrt{\gamma} = e^{3\alpha} \ ; \ \boxed{L_{kin} = V_0 \tfrac{6e^{3\alpha}}{N}\big(\dot{\beta}_+^2 + \dot{\beta}_-^2 - \dot{\alpha}^2\big).} \tag{2.25}$$

This is the form of the kinetic term for *all diagonal, Bianchi-A metrics and all have one minus sign in the kinetic term.* The supermetric thus has Lorentzian signature. This continues to hold for Euclidean signature spacetime metrics.

The corresponding momenta are given by,

$$p_\alpha = -\frac{12e^{3\alpha}}{N}\dot{\alpha} \ , \ p_\pm = \frac{12e^{3\alpha}}{N}\dot{\beta}_\pm \tag{2.26}$$

The potential term, $R(\gamma)$ is obtained as,

$$R(\alpha, \beta_\pm, n) = e^{-2\alpha}\left[\frac{1}{2}\left(n_1^2 e^{4\beta_+ + 4\sqrt{3}\beta_-} + n_2^2 e^{4\beta_+ - 4\sqrt{3}\beta_-} + n_3^2 e^{-8\beta_+}\right)\right. \tag{2.27}$$

$$\left. - \left(n_1 n_2 e^{4\beta_+} + n_2 n_3 e^{-2\beta_+ - 2\sqrt{3}\beta_-} + n_3 n_1 e^{-2\beta_+ + 2\sqrt{3}\beta_-}\right)\right] \ . \tag{2.28}$$

Finally, the Hamiltonian takes the form[3],

$$H = N\left[\frac{e^{-3\alpha}}{24V_0}\big(-p_\alpha^2 + p_+^2 + p_-^2\big) - V_0 e^{\alpha} R(\alpha, \beta_\pm)\right] := H_{kin} + H_{pot}. \tag{2.29}$$

The kinematical phase space is 6 dimensional, we have one first class constraint and physical phase space is thus 4 dimensional – the $H = 0$ constraint surface modulo the trajectories on it

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[2] In fact more is true: The signature of the supermetric continues to be Lorentzian even for the general gravitational phase space. This is related to the conformal factor problem of the Einstein-Hilbert action that complicates the corresponding (*Euclidean*) path integral. See ([6]).

[3] If a cosmological constant is included, then the term $+N\Lambda e^{3\alpha}V_0$ is to be added to H_{pot}.

generated by H. The physical space may be taken as being coordinatized by $\beta_\pm, p_\pm$, with no generator of evolution! This completes the canonical form of the dynamics of homogeneous diagonal, class A Bianchi spacetimes.

E. Examples

Let us focus on the two classic examples of models which also include the isotropic enhancement. These are the Bianchi-I and the Bianchi-IX models, $n_1 = n_2 = n_3 = 0,\ 1$ respectively. We also take $V_0 = 1$ for convenience.

Momentarily ignore how we come to the canonical system (2.29) and analyze the system just as a system whose Hamiltonian is a constraint, which is first class by definition. Let t denote an evolution parameter in terms of which the Hamilton's equations of motion are written as $\frac{d*}{dt} \approx N\{*, H\} + \{*, N\}H$. The second term vanishes on the constrained surface $\tilde{\Gamma}$.

Bianchi-I: The potential term vanishes. Equations of motion give

$$\dot{\alpha} = -\frac{Ne^{-3\alpha}}{12}p_\alpha\ ,\ \dot{\beta}_\pm = \frac{Ne^{-3\alpha}}{12}p_\pm\ ;\ \ \dot{p}_{\alpha,\beta_\pm}\ \propto\ NH\ \simeq\ 0\ ;\ H \simeq 0\ \Rightarrow p_\alpha^2 = p_+^2 + p_-^2\ .$$

The momenta are constants and satisfy the constraint condition. One of the solutions is: $p_{\alpha,\pm} = 0 \Rightarrow \alpha, \beta_\pm$ are constants. This is of course gives the Minkowski metric. This trivial solution is ignored below.

Thanks to the Lagrange multiplier N which can be a function of the phase space variables, we can choose different evolution parameters and simplify integration of the equations of motion. Two natural choices are: (i) $ds = Ne^{-3\alpha}dt$ ("Misner time"), (ii) $Ndt = d\tau$ ("synchronous/proper time"). These are related by $ds = e^{-3\alpha}d\tau$. In terms of s, the solutions are trivially obtained as, $\alpha(s) = -\frac{p_\alpha}{12}s + \alpha_0,\ \beta_\pm(s) = \frac{p_\pm}{12}s + \beta_{\pm,0}$. These are straight lines in the $\beta_\pm$ plane. Choosing integration constants (α_0, τ_0) suitably, we can arrange $ln(\tau) = -\frac{p_\alpha}{4}s$. On $\tilde{\Gamma}$, $p_\alpha = \pm\sqrt{p_+^2 + p_-^2}$. Choosing the *negative* sign, we see that $s \in (-\infty, +\infty) \leftrightarrow \tau \in (0, \infty)$. In terms of τ, the solution take the form: $\alpha(\tau) = \frac{ln(\tau)}{3}, \beta_\pm(\tau) = \frac{p_\pm}{\sqrt{p_+^2+p_-^2}}\frac{ln(\tau)}{3}$. We observe that evolution in s is complete while that in τ is incomplete.

From constrained theory perspective, we have 6 dimensional *kinematical phase space*, Γ, the 5 dimensional *constrained surface* $\tilde{\Gamma}$ and would have the space of gauge orbits (whole trajectories) or *reduced phase space* $\hat{\Gamma}$ which is 4 dimensional. There is *no* evolution in the reduced phase space i.e. going from one gauge orbit to another. The system with Hamiltonian constraint has just given us a smaller phase space with no dynamics.

This "frozen" affair can also be manifested in another way. Solve the Hamiltonian constraint strongly and set $p_\alpha = \pm\sqrt{p_+^2 + p_-^2} =: \tilde{H}$. By definition, Dirac observables are functions on Γ whose Poisson bracket with the constraint is weakly zero. For a generic $F(\alpha, \beta_\pm, p_\alpha, p_\pm)$, this condition becomes:

$$
\begin{aligned}
\partial_\alpha F &= \{F, \tilde{H}(p_\pm)\} = \partial_{\beta_+} F \partial_{p_+} \tilde{H} + \partial_{\beta_-} F \partial_{p_-} \tilde{H} \quad \Rightarrow \\
\partial_\alpha p_\pm &= 0 \quad , \quad \partial_\alpha \beta_\pm \; = \; \frac{p_\pm}{\sqrt{p_+^2 + p_-^2}} \; .
\end{aligned}
$$

Thus, the specific functions $\beta_\pm(\alpha), p_\pm(\alpha)$ *are* the 4 Dirac observables. These functions describe the evolution of the $\beta_\pm, p_\pm$ *relative to* α. This is also referred to as relational evolution or internal time (α) evolution. The variable α is not limited in anyway i.e. these evolution are also complete and completely internal to the phase space variables.

Thus, in this example we see that we can interpret the system in terms of *multiple external time* evolutions, some of which are complete and some incomplete. We can also interpret the system in terms of *multiple internal time* evolutions. The evolution relative to α, β_+, β_- are all complete[4].

By introducing a parameter t and *choosing* a parametrization $\alpha(t)$ we can of course convert the internal $\alpha-$evolution to an external $t-$evolution, in several ways. We may also construct a *space-time interpretation* by introducing $a_i(t) := e^{\beta_i(t)}$ where β_i are related to the $\alpha, \beta_\pm$ as the Misner parametrization and defining a line element as $ds^2 = -N^2 dt^2 + \sum_i a_i^2(t)(dx^i)^2$. Now the phase space evolution will appear as evolution of a homogeneous universe in terms of t! The arbitrariness of choice of t has been allowed for by introducing the arbitrary lapse function N^2. Now, $d\tau := N dt$ is indeed the *proper time* of an observer whose worldline has constant spatial coordinates x^i. The solutions discussed above, take the commonly presented

[4] Incidentally, also note that in general, solution of the Hamiltonian constraint $H \approx 0$ in the form $p_{..} = \tilde{H}$ can only be done *locally* (implicit function theorem) and thus such locally defined evolutions cannot be extended to a global evolution. The Bianchi-I is rather simple.

form (*the Kasner solution*),

$$ds^2 = -d\tau^2 + \tau^{2p_1}dx_1^2 + \tau^{2p_2}dx_2^2 + \tau^{2p_3}dx_3^2 \quad , \quad \sum_i p_i = 1 = \sum_i p_i^2 \ .$$

The constraints on the p_i imply that *all exponents cannot be positive, at least one must be negative.* This means that as $\tau \to 0$ two of the scale factors vanish while the third one diverges. The interpreted space-time metric immediately defines the space-time Riemann tensor squares, shear of the congruence of the vector field ∂_τ etc. The 3-Ricci scalar is zero, but the Riemann tensor square goes as τ^{-4}. The shear, σ^2 of the congruence of the vector field ∂_τ, goes as τ^{-2}.

The behavior scale factors show the anisotropic singularity as $\tau \to 0$. *Everyone of the gauge orbits* (except the Minkowski space-time) shows this "incompleteness" of an observer's worldline (these are also freely falling observers).

If the constructed space-time is viewed physically so that experiments/observations may be interpreted with this line element, then we have a observational handle on the underlying phase space dynamics. This much follows in this simple example. What happens for non-zero potential?

Bianchi-IX: The potential term takes the form,

$$H_{pot} = -N\frac{e^{\alpha}}{2}\left[e^{-8\beta_+} - 4e^{-2\beta_+}cosh(2\sqrt{3}\beta_-) + e^{4\beta_+}\{cosh(4\sqrt{3}\beta_-) - 1\}\right] \tag{2.30}$$

The dimensionalities of $\Gamma, \tilde{\Gamma}, \hat{\Gamma}$ all remain the same. However, both the external and the internal evolutions are much less explicit. No close form solution is know. The $\tau-$evolution is *chaotic*. Furthermore, now the $\alpha-$evolution is no longer monotonic in τ. The space-time interpretation of the universe corresponds to Kasner epochs of expansion/contractions along the different directions which themselves keep changing. This is a separate fascinating story, but is not our concern at present.

Inclusion of Λ: The inclusion of cosmological constant is done by adding the term $+\Lambda e^{3\alpha}$ to the Hamiltonian in equation (2.29). The vanishing of the constraint already shows that the $\Lambda e^{6\alpha}$ term becomes negligible as $\alpha \to -\infty$ and hence the behavior goes over to Kasner. For α positive and large, the anisotropy momenta are negligible and anisotropies get suppressed. For negative Λ, the anisotropy momenta can cancel the $-|\Lambda|$ contribution implying a re-collapse. In addition, with positive Λ, when Λ begins to dominate it also discourages entering the small volume regions. For negative Λ this does not happen. Thus, for $\Lambda > 0$, the $\tau \to 0$ singular, oscillatory behavior is unaffected while for $\tau \to \infty$, oscillations stop and universe enters de Sitter expansion. For $\Lambda < 0$, the τ behavior has epochs of Λ dominance, but oscillations do not stop either in the early or in late epochs.

So far we do not see any "problem of time" apart from absence of any preferred choice of evolution parameter and what to make of (in)complete evolutions. Furthermore, thanks to the Hamiltonian constraint, *any* evolution is a gauge transformation. Even if all gauge orbits are incomplete (in the τ evolution), but *why should that be* physically *relevant?*

The relevance comes from the *space-time reconstruction* seen above. We could view the entire exercise as a possibility of *interpreting* a Hamiltonian system in terms of space-time geometry. The fact that space-time geometry, thanks to covariance has built-in arbitrary coordinates and to accommodate that, the Hamiltonian system should be made as a constrained system with Hamiltonian as a constraint. This promptly regards generic evolution parameters as "coordinates" and the space-time geometry takes care of the *physical* but observer dependent time – the timelike intervals ($-\Delta s^2$). Only after "time" born this way can a "problem of time" arise. Note that a space-time reconstruction is mandatory for interpretation of observations/experiments and a general relativistic space-time automatically gives observer dependent time intervals.

This also brings out the point/possibility that space-time singularities arise when one tries to reconstruct a space-time from a phase space. Even if the phase space dynamics is free of pathologies, the reconstruction can potentially generate *interpretational pathologies* (eg, causality, determinism, geometry-matter interaction). The singularity theorems of the space-time view, identify what situations can have such problems. (Hence study the phase space dynamics in those contexts - cosmology and collapse!) Quantum framework, tied to a phase space, is an obvious candidate for an improved reconstruction. The general classical analysis helps expose the challenge of reconstruction.

Having noted the space-time reconstruction step in an interpretation of a Hamiltonian dynamics, let see what issues arise at the quantum level. For this, we consider the simpler, minisuperspace quantization.

3. MINISUPERSPACE QUANTIZATION

There are different methods of quantization, given a classical system formulated in terms of an action in the Lagrangian or Hamiltonian form. The canonical quantization (in terms of the metric variables) is known as the Wheeler-de Witt quantization. There is technically different, loop quantization based on connection variables. There is also the "path integral quantization" (phase space/configuration space). Here are brief comments on the WdW.

Wheeler-de Witt (WdW) quantization:

Quite generally, quantization of a classical theory with first class constraints, has two routes to proceed: (a) Promote the classical theory to usual canonical quantization constructing a *kinematical Hilbert space*, H_{kin}, ignoring the constraints and propose that the *physical states* are solution are those that are annihilated by the constraints; or (b) solve the constraints classically to get a unconstrained theory on $\hat{\Gamma}$ and quantize this in the usual manner. The former is called *Dirac quantization* and the latter is called *reduced phase space quantization*. The latter may look "cleaner", however (i) it is much harder to solve the constraints completely to get $\tilde{\Gamma}$ and (ii) even if the $\tilde{\Gamma}$ is obtained explicitly, it will have complicated topology and/or global structure as a symplectic manifold, making its quantization non-trivial. We will see the issues that arise in the Dirac quantization implicit in the WdW procedure.

Time also plays a central role in a canonical quantization procedure. In the *non-relativistic* context, time development of a quantum state is an explicit postulate, be it in any of the Schrödinger/Heisenberg/Dirac pictures. This "time" is the usual time recorded by a clock and is not part of the quantum system. Causality is also simple time ordering and the inner product defining the Hilbert space is *independent of time*. In a special relativistic context, we already have clock time dependent on the observer's state of motion in the sense that different observers in relative uniform motion have different clock rates. Naive generalization of wavefunction fails due to lack of inner product and a restriction of the space of solution of the relativistic wave equations needs to be done. This restriction is conditioned on a *a choice of time* with respect to which positive frequency/energy solutions have to be identified. Physically this arises due to the existence of anti-particles and the natural quantum framework becomes that of a quantum field. The Hilbert space is constructed as *a Fock space* (See Wald [2, 7]). Fortunately, the process does not give inequivalent Hilbert space for the proper times of observers who are in relative uniform motion. Situation changes once *accelerated observers* are involved and we have inequivalent Hilbert spaces constructed by two such observers or relative to their corresponding proper times (Unruh effect in Minkowski spacetime). In all these cases too, the inner product is *independent of the corresponding time variable.* With gravity, the situation is far more complex and we can expect inequivalent quantization even before getting into technical details. We see a potential "problem of time" as a possible obstruction to construction of a Hilbert space of physical state to begin with *before* facing the interpretational issues.

Keeping these in mind let us return to quantization of the a finite dimensional phase space with a Hamiltonian constraint.

The classical mini-superspace system has a configuration space Q, with coordinates $q^\alpha, \alpha = 1, \ldots, n$ and with the action of the form (Halliwell Lectures [8]),

$$S[q^\alpha, N] = \int_0^1 dt\, N(t) \left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta - U(q)\right] \ , \ f_{\alpha\beta} \text{ is an indefinite metric;} \tag{3.1}$$

$$p_\alpha = \frac{1}{N} f_{\alpha\beta}\dot{q}^\beta \ , \ H_{can} = NH \ \ , \ \ H = \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) \ \simeq \ 0 \ . \tag{3.2}$$

$$0 = H(q^\alpha, -i\partial_\alpha)\Psi(q) \ , \ \text{Dirac quantization.} \tag{3.3}$$

In writing the last equation, we have already presumed some formal Hilbert space of functions of the coordinates. Since the metric depends on q^α, there is an ordering issue and is usually resolved by demanding that the equation should be covariant under change of q^α ("field redefinitions"). This leads to,

$$\hat{H} = -\frac{1}{2}\nabla^2 + U(q) \ , \qquad \text{Operator on } \mathcal{H}_{kin} := L^2(Q, \sqrt{|det(f_{\alpha\beta})|}d^n q). \tag{3.4}$$

We have suppressed a possible Ricci scalar, $R(f)$ term which does not concern us at present. The Laplacian is defined by the metric on the superspace. Dirac quantization consists of selecting solutions of "$\hat{H}\Psi(q) = 0$" as physical states. This is a formal statement since typically, the solutions will *not* be in the kinematical Hilbert space. Instead one proceeds as follows[5].

[5] It is useful to keep in mind the explicit example of Bianchi-I, with/out massless scalar ϕ. Then the Q is 4 dimensional with coordinates $(\alpha, \beta_\pm, \phi)$ and momenta are $(p_\alpha, p_{\beta_\pm}, p_\phi)$. The metric $f^{\alpha\beta} = e^{-3\alpha} diag(-\frac{1}{12}, 1, 1, 1)$ and $U(q) = 0$ There is no ordering issue and the $\mathcal{H}_{kin}$ can be taken to be $L^2(\mathbb{R}^4, d^4q)$.

View the formal expression as a partial differential equation on Q. This is an hyperbolic equation since the supermetric has Lorentzian signature. This is the *Wheeler-de Witt* (WdW) equation. There are of course infinitely many solutions. What do they represent? Does any solution "correspond" to our (classical) universe space-time? This is a problem of interpretation, choice of solution(s) and extracting observable implications.

We separate the two situations one without scalar matter and one with scalar matter to appreciate different facets of the difficulties. We will use the context of the Bianchi dynamics.

No matter degrees of freedom: The WdW equation takes the form, $(\partial_\alpha^2 - 12\partial_+^2 - 12\partial_-^2)\Psi(\alpha, \beta_\pm) = 0$. This has the same form as the massless Klein-Gordon equation in (2+1) dimensions. Here we have a conserved current, and a naturally singled out "time direction" α. The usual Klein-Gordon inner product is positive on the subspace positive frequency solution of the WdW equation. Hence, the space of positive frequency solutions constitutes the Hilbert space of physical states with the Klein-Gordon inner product. The conserved current ensures that this physical inner product is $\alpha-$independent.

Scalar matter included: The WdW equation takes the form, $(\frac{1}{12}\partial_\alpha^2 - \partial_+^2 - \partial_-^2 - \partial_\phi^2)\Psi(\phi, \alpha, \beta_\pm) = 0$. This has the same form as the massless Klein-Gordon equation in (3+1) dimensions. Here too we have analogous conserved current, but now *choose* ϕ as the "time direction". However, since the "space" Laplacian operator is no longer negative definite, we can't guarantee real frequencies and there is no meaningful split of positive/negative frequencies. The restriction to ϕ-constant surfaces does not give a positive definite integral. One can proceed differently though. Consider a constant ϕ hypersurface and restrict the solutions of WdW equation to this hypersurface. We have the induced inner product, inherited from the $\mathcal{H}_{kin}$ on these functions. Choose appropriate boundary conditions on the hypersurface so as to define the Laplacian as a self-adjoint operator, $\hat{\Theta}$. Restrict to the subspace where the operator is non-negative and take its positive square root, $\sqrt{\hat{\Theta}}$. The quantization is completed by taking the Schrödinger equation as: $i\partial_\phi\Psi(\phi, \alpha, \beta_\pm) = \sqrt{\hat{\Theta}}\Psi(\phi, \alpha, \beta_\pm)$. The physical states are identified as solutions of non-negative spectral subspace of the solution of the WdW equations, restricted to constant-ϕ hypersurface. The construction gives a unitarily equivalent physical Hilbert space if the $\phi-$constant hypersurface is changed.

"Deparametrization": Notice that this quantization can also be understood as reduced phase space quantization where, the classical Hamiltonian constraint is solved as $p_\phi = \tilde{H} :=$

$\sqrt{-\frac{1}{12}p_\alpha^2 + p_+^2 + p_-^2}$ and promoting the $\tilde{H}$ to a well defined operator. In this case of course there is no WDW equation, space of solutions, kinematical and physical inner products. Just one inner product to be introduced to define the square root operator.

We have taken the explicit example of the simplest case of Bianchi-I. The kinetic term in all Bianchi models is the same. The Lorentzian signature of the kinetic term thus persists. The three options detailed above are available in all cases though the level of explicit forms varies. I will not try to articulate precisely the "problem of time" any further.

Wavefunction of the universe:

Let us assume that we have managed to get a quantum theory of mini-superspace with some physical inner product and with a Schrödinger equation and let us denote the clock symbolically by ϕ. What next?

We are faced with infinitely many solutions $\Psi(\phi, q)$. What criteria can we introduce to identify "sensible" solutions? Halliwell [8] states two minimal conditions: (i) the wave function or the appropriate probability distribution (eg Wigner distribution) should be peaked about correlations that describe the classical configurations from which space-time reconstruction can be done (in short a classical universe space-time); (ii) There are several such classical space-times and these should be well separated i.e. not interfere or their superposition should decohere. The search for meeting these criteria uses the WKB intuition of classically allowed (oscillatory) and disallowed (exponentially growing/vanishing) functional dependence.

At this stage, we *suspend* the mathematical search for inner product etc. Following Halliwell, we look for solutions of the WdW partial differential equation (say even without scalar matter) which are of WKB form (recall eikonal approximation in optics). The rays are in the superspace and thus correspond to space-time metric (spatial metric to begin with and space-time after reconstruction), the parameter along the rays being an external evolution parameter. The problem of "initial conditions" for the universe now refers to the choice of a wavefront. The idea and hope is that *if* we could choose a unique eikonal solution, then the rays will give the possible universes "beginning" at the chosen wavefront. If we can have a probability measure, we may estimate the probability for a particular history of the universe, thereby providing a *quantum explanation of origin and evolution of our universe.* Thus we need a proposal for a wavefunction solving the WdW equation in an eikonal approximation and a probability measure on the initial conditions on the rays. This is very different from the earlier mathematical discussion. It is limited though from the perspective of constructing the full space of physical solutions.

There are several proposals for the candidate wavefunctions, the most common being (a) Hartle-Hawking "no boundary" proposal and (b) Vilenkin's "tunneling" wave function. The former, gives an explicit function using Euclidean path integral while the latter requires the solution to have only "out-going" rays or universe emerging from nothing.

"Applications":

One can go beyond the minisuperspace by including the inhomogeneous fields in a perturbative manner: $0 \simeq H_{mini} + H_{inhomo}$. In these perturbative corrections, the unperturbed wavefunction is taken to be the chosen wavefunction of the universe. The wavefunction is taken in a WKB form with leading term reflecting the homogeneous background spacetime and the semiclassical correction terms reflecting the quantized perturbative modes emerging from the vacuum. Apparently, the above proposed wavefunctions of the universe provides the justification for the Bunch-Davies vacuum for the perturbation.

For details, I refer you to [6, 8].

In any other quantization, similar steps have to be followed. These are also followed in LQC, though the details differ.

In summary, to my mind, the classical canonical analysis puts the space-time differential equations of Einstein into Hamilton's equations, with first class constraints, in a phase space. The physical degrees of freedom and the arbitrary Lagrange multipliers present in the 4-metric are cleanly separated. One may view the phase space dynamics autonomously and then reinterpret it as a space-time. This can be done in multitude of ways and could harbor interpretational pathologies in the space-time, over and above those which may be encountered in phase space. A spacetime reconstruction is essential because all experiments/observations are phrased in terms of "rods and clocks" (spatial/temporal intervals).

If the hurdles in constructing a quantum framework for the physical degrees of freedom either by reduced phase space or by Dirac quantization are overcome, then the observables which are built from the 3-geometry will have expectation values in some quantum states. These will now enter the reconstructed space-time and determine the space and time intervals measured by instruments. The quantum fluctuations should then be visible in these measurements. There are several examples of mini-superspaces and reduced phase spaces that came up during the meeting. These could be tested for interpretation.

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