JT Gravity, Dilaton Gravity Models and The Problem Of Time

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A General Introduction

- Gravity is notoriously difficult to quantize, owing to a host of technical and conceptual issues.
- In dimensions, $d \geq 4$, especially, the technical difficulties like the UV divergence are so severe that the quantization of these systems become rather cumbersome.
- This also suggests that at lower dimensions where Gravity has zero degrees of freedom, the UV divergences are somewhat tamed if not completely absent.
- Many conceptual issues such as
  - Criteria for acceptable wavefunctions
  - Meaning of the wavefunction
  - Observers and Observables
  - Notion of time etc.,

  however still remain in these lower dimension theories.

- In this scenario Jackiw-Teitelboim Gravity (JT) gravity becomes especially useful in trying to tackle the conceptual issues.
- JT gravity, is a simple model of 2D gravity, involving metric and a scalar field called dilaton with the action

$$S_{\mathrm{JT}} = \frac{1}{16\pi G}\left(\int d^2x\,\sqrt{-g}\,\phi(R-2) - 2\int_{bdy}\sqrt{-\gamma}\phi K\right)$$

  where ϕ is the dilaton.
- More generally JT gravity is a specific example of dilaton gravity models with a generic potential of the dilaton, $U(\phi)$ whose action is given by,

$$S_{\mathrm{U}} = \frac{1}{16\pi G}\left(\int d^2x\,\sqrt{-g}\,(\phi R - U'(\phi)) - 2\int_{bdy}\sqrt{-\gamma}\phi K\right)$$

  When the potential is quadratic, one gets the JT gravity.

- JT gravity has no propagating bulk degrees of freedom and only boundary degrees of freedom are present.
- However what makes JT gravity so useful is the fact that one can calculate the no-boundary wavefunction, also called the Hartle Hawking wavefunction, in path integral method, exactly in this model.
- The resulting wavefunction is a function of boundary degrees of freedom, namely- the size of the universe and the value of the dilaton at the boundary.
- However for generic dilaton gravity models the path integral evaluation of no-boundary wavefunction is not straightforward.

Problem of Time

- Coordinate time is merely a label for events in spacetime. Diffeomorphism invariance of gravity theory means that any coordinate time is not special.
- Nothing physical can depend on the choice of coordinate time, t.
- The invariance of physics in the quantum theory means that the quantum wavefunction is invariant under and time and space reparametrizations.
- More precisely, the quantum wavefunction cannot depend on coordinate time.
- However, the quantum wavefunction encodes the dynamics of the spacetime and must change under the evolution of “Physical time".
- **Problem of Time** refers to finding a degree of freedom internal to the system which can satisfactorily play the role of physical time.

What do we expect from a good physical clock?

At each value of this physical clock, we should have

- A set of acceptable states,
- A well-defined inner product/norm on the Hilbert space,
- A good set of observables/calculables which can act on the Hilbert space,
- Expectation values for these observables, that would tell us about the properties of the universe as it evolves.

Goal of the Talk

- Introduction
- Classical Solutions in the presence of dilaton potential
- Canonical Quantization in JT gravity
- Matching It with the path integral answer
- Wavefunction In presence of dilaton potential
- Exponential Correction Potential
- Hartle Hawking State in General Theories
- Connection With matrix model
- Open Questions

Classical Solutions

The dilaton gravity model with a generic potential $U(\phi)$ is described by the action,

$$S_{\mathrm{U}} = \frac{1}{16\pi G}\left(\int d^2x\,\sqrt{-g}\,(\phi R - U'(\phi)) - 2\int_{bdy}\sqrt{-\gamma}\phi K\right)$$

The equations of motion are,

$$R = U''(\phi)\ ,\ \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi - \frac{1}{2}g_{\mu\nu}U'(\phi) = 0$$

Equations of motion imply the existence of the constant of motion,

$$M = \nabla^\mu\phi\nabla_\mu\phi + U(\phi)$$

and a killing vector,

$$\xi^\mu = \epsilon^{\mu\nu}\nabla_\nu\phi$$

with the norm $\xi^\mu\xi_\mu = U(\phi) - M$. Thus ξ is spacelike for $U(\phi) > M$.

Then considering the situations when $U(\phi) > M$ one can write down an ansatz for the metric where the killing symmetry is manifest. So,

$$ds^2 = -dt^2 + a(t)^2 dx^2, \phi = \phi(t)$$

Solving the Einstein equations one obtains,

$$a(t) = \sqrt{U(\phi(t)) - M}$$

The metric can then be recast into the form,

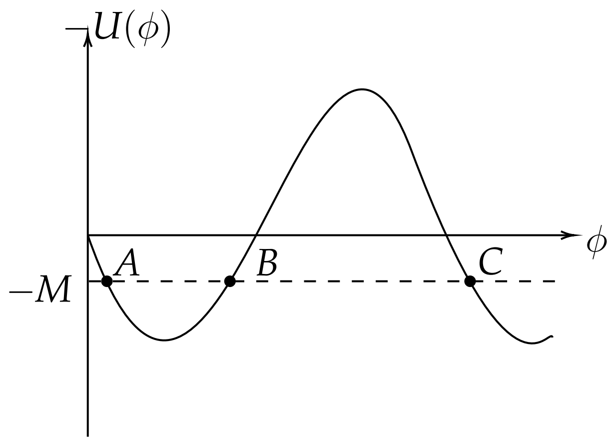
$$ds^2 = -\frac{dr^2}{U(r) - M} + (U(r) - M)dx^2, \ \phi = r$$

- The metric becomes singular at $U(r) = M$ but if the Ricci scalar is finite then we will loosely refer to this location as horizon.
- Unlike JT we now can have a genuine curvature singularity.
- The metric can easily be extended into situations where $U(\phi) < M$.

The conserved constant can be rewritten as,

$$\dot{\phi}^2 - U(\phi) = -M \tag{1}$$

This is akin to a particle moving in potential $-U(\phi)$ with total energy $-M$. Let $M > 0$.



JT Example

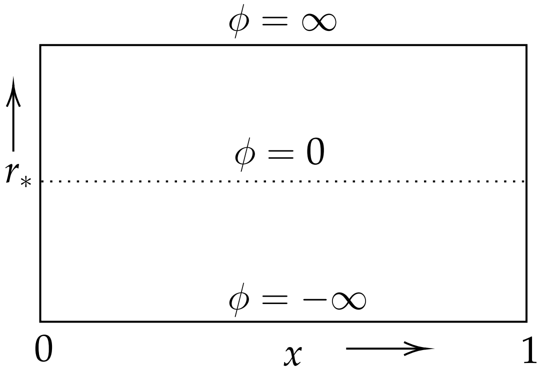
Let us now consider the JT example where $U(r) = r^2$. Depending on the sign of M one will end up with different spacetimes. For example if $M < 0$ we get the bounce sector. For $M > 0$ we get the big bang/big crunch sector.

We will often consider these two spacetimes with an orbifold described by the metric,

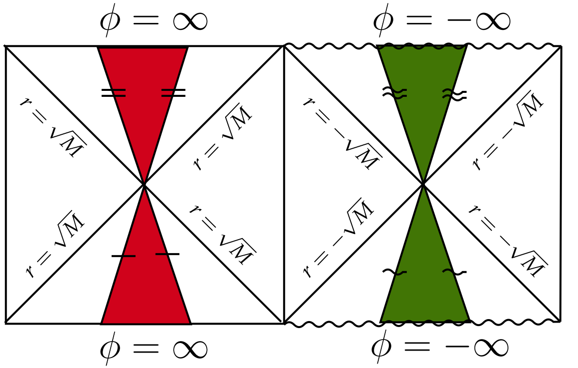
$$ds^2 = -\frac{dr^2}{U(r) - M} + (U(r) - M)\frac{dx^2}{A^2}, \ \phi = r.$$

There is also one more classical solution possible. In this case one considers a big bang/big crunch type spacetime with alternating cosmological and black hole regions but with endpoints identified.

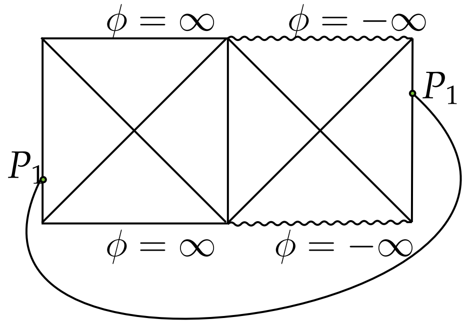
Ref: J.Held, H.Maxfield: The Hilbert space of de Sitter JT: a case study for canonical methods in quantum gravity,2024



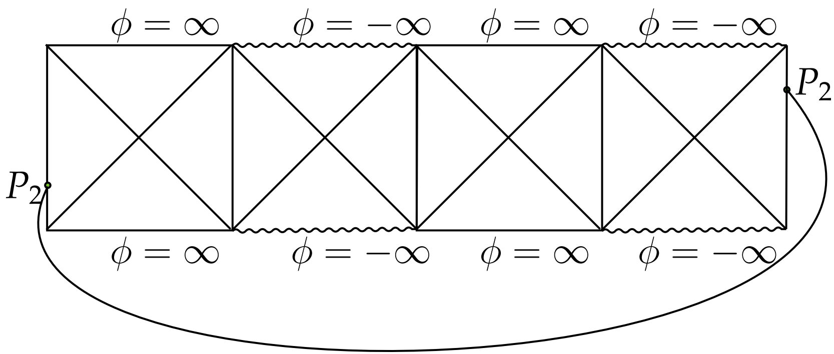
(a)Bounce Sector



(b)Big bang / Big crunch Sector


$\phi = \infty$
$\phi = -\infty$
P_1
P_1
$\phi = \infty$
$\phi = -\infty$

$n = 1$


$\phi = \infty$
$\phi = -\infty$
$\phi = \infty$
$\phi = -\infty$
P_2
P_2
$\phi = \infty$
$\phi = -\infty$
$\phi = \infty$
$\phi = -\infty$

$n = 2$

Phase Space Analysis

Let us now try to understand the phase space of JT gravity. Recall the problem of motion of a particle. From the on shell action one gets the pre-symplectic current θ and sympletctic form to be,

$$\theta = p\delta q, J = -\delta\theta = \delta q \wedge \delta p$$

A similar analysis for JT following CW (1987) gives the symplectic form to be,

$$J = \frac{1}{2}\delta A \wedge \delta m \quad , m = \frac{M}{A^2}$$

In terms of Poisson Brackets

$$\{M, \frac{1}{2A}\} = 1$$

It turns out that even for general potential the above relation holds true.
Ref: C. Crnkovic, E.Witten- Covariant description of canonical formalism in geometrical theories, 1987

Constraint Analysis

In the case of dilaton gravity models the constraints are first class. That is given,

$$
\begin{aligned}
\mathcal{H} &= 2\pi_\phi \pi_{g_1}\sqrt{g_1} - \left(\frac{\phi'}{\sqrt{g_1}}\right)' - \frac{\sqrt{g_1}U'(\phi)}{2} \\
\mathcal{P} &= 2g_1\pi'_{g_1} + \pi_{g_1}g'_1 - \pi_\phi\phi'
\end{aligned}
$$

and,

$$
\begin{aligned}
Q_H &= \int \sqrt{g_1(x)}\epsilon_H(x)\mathcal{H}(x)dx \\
Q_P &= \int \epsilon_P(y)\mathcal{P}(y)dy
\end{aligned}
$$

it can be shown that,

$$\{Q_P, Q_P\} = Q_P, \{Q_P, Q_H\} = Q_H, \{Q_H, Q_H\} = Q_P$$

Canonical Quantization

We will now take our first step towards quantization. Our quantization procedure will apply to those spacetimes where one can define a spacelike hypersurface at a constant value of ϕ. This will be the case for big bang/crunch and bounce sector.
Given such a hypersurface we can use the invariance of wavefunction under the spatial reparametrizations to argue that the wavefunction can only depend on the length of the spatial hypersurface l and the value of the dilaton at the hypersurface ϕ.
Using spatial reparametrization one gets the Hamiltonian constraint to be,

$$\mathcal{H} = 2\sqrt{g_1}\pi_{g_1}\pi_\phi - \sqrt{g_1}\phi$$

Now $l = \sqrt{g_1}$. This then leads to the following equation for Ψ

$$\left(\partial_l\partial_\phi - \frac{1}{l}\partial_\phi + l\phi\right)\Psi(l,\phi) = 0$$

This is the Wheeler de Witt equation.

An Algebra

The operators we will be considering are given by,

$$\hat{G}_1 = \partial_\phi^2 + l^2, \hat{G}_3 = (-i)(l\partial_l - \phi\partial_\phi), \hat{M} = \partial_l^2 + \phi^2$$

The various commutators between $\mathcal{H}$ and $\hat{G}_1, \hat{M}, \hat{G}_3$ can be worked out to be,

$$[\mathcal{H}, \hat{G}_1] = 0, \;\; [\mathcal{H}, \hat{M}] = -\frac{2}{l^2}\mathcal{H}, \;\; [\mathcal{H}, \hat{G}_3] = 0.$$

The commutators between $\hat{G}_1, \hat{M}, \hat{G}_3$ are found to be,

$$[\hat{G}_1, \hat{M}] = -4i\hat{G}_3, \;\; [\hat{G}_1, \hat{G}_3] = 2i\hat{G}_1, \;\; [\hat{M}, \hat{G}_3] = -2i\hat{M}$$

By working in the subspace where $\mathcal{H} = 0$ we can set the second commutator $[\mathcal{H}, \hat{M}]$ to zero. And thus $\hat{G}_1, \hat{M}, \hat{G}_3$ form an algebra in this subspace.

Canonical Quantization-II

Let us now focus on the Eigenfunctions of $\hat{M}$ in the physical subspace $\mathcal{H} = 0$. They take the form,

$$\Psi = e^{\mp il\sqrt{\phi^2 - M}}$$

$-$: Expanding, $+$: Contracting. In general one can write,

$$\Psi = \int_{-\infty}^{\infty} dM\rho(M)e^{-il\sqrt{\phi^2-M}}$$
$$\Psi = \int_{-\infty}^{\infty} dM\tilde{\rho}(M)e^{il\sqrt{\phi^2-M}}$$

We will often work with the wavefunction,

$$\hat{\Psi} = \frac{1}{l}e^{\mp il\sqrt{\phi^2-M}}$$

which satisfies the equation,

$$\left(\partial_l\partial_\phi + l\phi\right)\hat{\Psi}(l,\phi) = 0$$

Phase Space Analysis-A Check

Before moving on let us do a check.
In JT we have two parameters M, A. And as we saw earlier these two are canonically conjugate.
More concretely,

$$[M, \frac{1}{2A}] = i$$

This suggests that a generic wavefunction in some limit should reduce to the form,

$$\Psi \sim e^{\pm i \frac{M}{2A}}$$

We obtained the wavefunction to be,

$$\Psi = e^{\mp il\sqrt{\phi^2 - M}} = e^{\mp il\phi} e^{\pm i \frac{M}{2A}}, \; A = \frac{\phi}{l}$$

Thus the canonical quantization gives us a wavefunction that has the desired property.

WKB Analysis- A Further Check

Let us do another check of the wavefunction that we constructed as solutions of WDW equation using WKB analysis. In this analysis we will consider

$$\hat{\Psi} = e^{S}$$

and we will expand S in powers of G.

$$S = \frac{S_{-1}}{G} + S_0 + S_1 G + \cdots$$

The WDW equations with explicit factors of G reads, $(8\pi = 1)$

$$G^2 \partial_l \partial_\phi \hat{\Psi}(l, \phi) + l\phi \hat{\Psi}(l, \phi) = 0$$

Using the expansion one gets for S_{-1}

$$\partial_l S_{-1} \partial_\phi S_{-1} + l\phi$$

whose solution gives

$$S_{-1} = -il\phi$$

Going one step further we obtain the following value

$$S_0 = -\frac{1}{2}\log(l\phi) + \log(g(\eta)), \quad \eta = \frac{\phi}{l}$$

where $g(\eta)$ is some arbitrary function of η. The higher order terms S_1, S_2 and so on are given as functions of $g(\eta)$.
To match with the earlier solution we will take

$$g(\eta) = \sqrt{\eta}\int dM\rho(M)e^{iM\frac{1}{2\eta}}$$

The wavefunction then reads

$$\hat{\Psi} = \frac{1}{l}e^{-il\phi}\int dM\rho(M)e^{iM\frac{l}{2\phi}}$$

which matches with the earlier solution in the limit $M \ll \phi^2$.

The WKB expansion just described is valid in the regime

$$l\phi \gg 1, \frac{l}{\phi}\text{held fixed}$$

The limit where the matching works on the other hand is given by

$$1 \ll l\phi \sim \frac{\phi^2}{M} \ll \left(\frac{\phi^2}{M}\right)^2$$

This procedure can easily be extended to cases with general potentials where the analogous limit for WKB expansion is given by

$$l\sqrt{|U(\phi)|} \gg 1, \frac{l}{\sqrt{|U(\phi)|}}\text{held fixed}$$

DILATON AS A CLOCK

- The gauge invariant quantities in the theory are the dilaton and the length of the boundary.
- In classical solutions, the dilaton field is a monotonic function of coordinate time. Thus, in the full quantum theory, dilaton can be chosen to be a clock.
- Dilaton as a clock has the advantage that it is a locally measurable quantity.
- Classical limit corresponds to

$$G \to 0 \quad \text{or equivalently} \quad \phi \to \infty$$

  In this talk, G is scaled away by taking $\frac{\phi}{8\pi G} \to \phi$.
- With Dilaton as a clock, the classical limit corresponds to "Late times".
- We will be interested in asking questions like
  - Whether norm of a state is constant in time.
  - Whether a classical limit arises at late time, i.e. at large ϕ.

Inner product

- In terms of the variables $u = l^2, v = \phi^2$, the modified Hamiltonian constraint equation is a massive Klein-Gordon equation with ϕ, l being null directions.
- This massive Klein gordon equations has exact solutions. We will come to it later.
- The natural inner product to consider is the Klein-Gordon inner product. The integral is evaluated at a particular value of the dilaton.

$$\langle \hat{\Psi}, \hat{\Psi} \rangle = \int_0^\infty dl \; i(\hat{\Psi}^* \partial_l \hat{\Psi} - \hat{\Psi} \partial_l \hat{\Psi}^*),$$

- Further need to impose the condition that the states have a conserved and finite norm.

$$\begin{aligned} \text{Conservation} &\Rightarrow \partial_\phi \langle \hat{\Psi}, \hat{\Psi} \rangle = 0 \\ &\Rightarrow \mathcal{C}_N \equiv i(\hat{\Psi} \partial_\phi \hat{\Psi}^* - \hat{\Psi}^* \partial_\phi \hat{\Psi}) = 0 \quad \text{at } l = 0, \infty \end{aligned}$$

- At late times, ϕ large, if Ψ has compact support in M, then the KG norm becomes L_2 norm for each of the branches,

$$\langle \hat{\Psi}, \hat{\Psi} \rangle \simeq 2\phi \int_0^\infty dl\, \hat{\Psi}^* \hat{\Psi}$$

  and thus is positive definite.
- At earlier times one must check the probability density on a case by case basis.
- The inner product between two normalized wavefunction is defined to be,

$$\langle \hat{\Psi}_1, \hat{\Psi}_2 \rangle = \frac{1}{2} \int_0^\infty dl i (\hat{\Psi}_1^* \partial_l \hat{\Psi}_2 - \hat{\Psi}_2 \partial_l \hat{\Psi}_1^*) + \frac{1}{2} \int_0^\infty dl i (\hat{\Psi}_2^* \partial_l \hat{\Psi}_1 - \hat{\Psi}_1 \partial_l \hat{\Psi}_2^*)$$

- The conservation of the inner product puts further constraints on $\rho_i(M), i = 1, 2$.

Expectation values

- All functions of the length of the boundary and its conjugate momentum are guage invariant quantities.

- Moments $\langle l^n \rangle$ of the length to be given by

$$\langle l^n \rangle = \int dl\, l^n\, p(l, \phi)$$

$$p(l, \phi) = i(\hat{\Psi}^* \partial_l \hat{\Psi} - \hat{\Psi} \partial_l \hat{\Psi}^*)$$

- For the conjugate momentum we have

$$\langle \pi_l^n \rangle = -\frac{1}{2^n} \int dl \hat{\Psi}^* (-i(\overrightarrow{\partial} - \overleftarrow{\partial}))^{n+1} \hat{\Psi}.$$

- Finally we can also calculate the expectation values of M.

$$\langle \hat{M} \rangle = \frac{1}{\mathcal{N}}(-\langle \pi_l^2 \rangle + \langle \phi^2 \rangle)$$

Constraints on ρ

The constraints on ρ come from the requirements that the norm should be finite and conserved. The analysis is rather involved. Here we just give a particular method which ensures that the norm is conserved.

- $\rho(M)$ should have compact support.
- If the support is over negative values of M, the norm is conserved everywhere. If the support is compact and over positive values of M, say $[0, M_0]$, then for $\phi^2 > M_0$ norm will be conserved.
- $\rho(M)$ should be real or have an M independent phase.

This ensures that

$$\hat{\Psi}^* \partial_\phi \hat{\Psi} = \hat{\Psi} \partial_\phi \hat{\Psi}^*$$

and $\mathcal{C}_N$ vanishes.

Finiteness of norm requires that in addition to above we must have,

$$\int \rho(M)dM = 0$$

and there can be no growing mode in $\hat{\Psi}$.

Further, different branches should sufficiently decohere. This means the interference between various branches should be sufficiently small. This can be achieved by working with a coefficient function that is compact.

Finally for two normalizable wavefunctions to have conserved inner product the respective coefficient functions should either be in phase or out of phase.

A Delta Function Example

- Consider the ρ
$$\rho = \delta(M + M_1) - \delta(M + M_2)$$
- Then the wavefunction is given by
$$\hat{\Psi} = \frac{1}{l}\left(\exp\left(-il\sqrt{\phi^2 + M_1}\right) - \exp\left(-il\sqrt{\phi^2 + M_2}\right)\right)$$
- The probability density is positive definite.
- The norm is found to be
$$\langle \hat{\Psi}, \hat{\Psi} \rangle = \pi |M_1 - M_2|$$
- All moments $\langle l^n \rangle$ diverge for $n > 1$. However the moments of π_l and $\hat{M}$ are finite.
$$\langle \hat{M} \rangle = -\frac{1}{2}(M_1 + M_2), \Delta M = \frac{|M_1 - M_2|}{2\sqrt{3}}$$

Infinite Number of Solutions

Take a one-parameter family of solutions labeled by M_1 where

$$\rho_{M_1}(M) = \delta(M + M_1) - \delta(M + M_1 - a)$$

Here a is held fixed and the family is obtained by taking M_1 to have values in the range $M_1 \in (a, \infty]$. The corresponding solutions are given by

$$\Psi_{M_1}(l, \phi) = e^{-il\sqrt{\phi^2 + M_1}} - e^{-il\sqrt{\phi^2 + M_1 - a}}$$

The wavefunction is a sum of two eigenvectors of $\hat{M}$ with eigenvalues $-M_1, -(M_1 - a)$. Hence different wavefunctions will be linearly independent of each other. Thus we can create an infinite family of solutions with finite and conserved norm.
These solutions can also be made orthogonal.

Theta Function

Coefficient function is given by

$$\rho(M) = \begin{cases} 1 & -M_0 \leq M \leq 0 \\ -1 & -2M_0 \leq M \leq -M_0 \end{cases}$$

In this case the wavefunction is given by,

$$\hat{\Psi} = \frac{e^{-il\sqrt{M_0+\phi^2}}\left(4+4il\sqrt{M_0+\phi^2}\right) + e^{-il\sqrt{2M_0+\phi^2}}\left(-2-2il\sqrt{2M_0+\phi^2}\right)}{l^3} + \frac{e^{-il\phi}(-2-2il\phi)}{l^3}$$

- The probability density is negative at early times and positive at late enough times.
- Norm is finite, positive and conserved.
- $\langle l \rangle, \langle l^2 \rangle$ are finite but higher moments diverge.

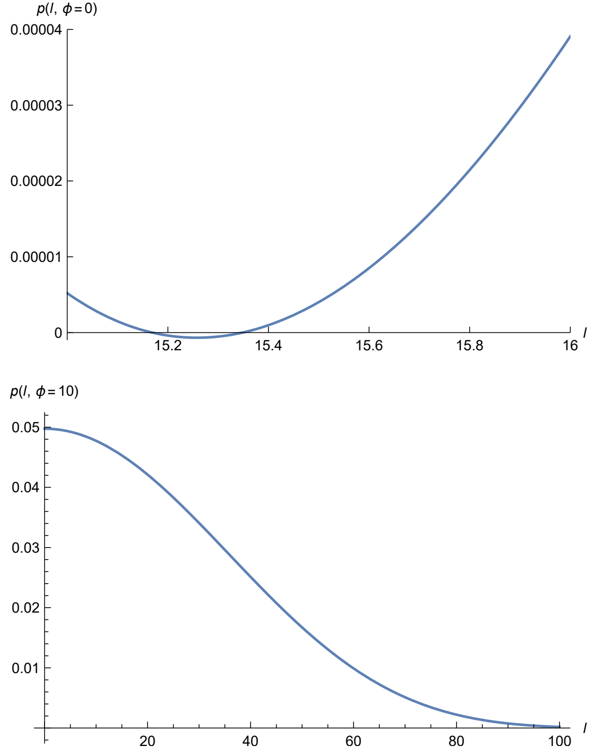


Figure. We have set $M_0 = 1$ in both the plots.

Wave Packet Example

- The wavefunction is given by

$$\hat{\Psi} = \frac{1}{l}\int_{-(\Delta+M_0)}^{\Delta-M_0} dM\,\rho(M)\; e^{-il\sqrt{\phi^2-M}}$$

with

$$\rho(M) = 2\sin((M+M_0)x_0)\,e^{-\frac{(M+M_0)^2}{2\sigma}},\; M \in [-\Delta+M_0, \Delta-M_0]$$

The wavefunction can be rearranged to give,

$$\hat{\Psi} = \frac{2}{l}\int_{-\Delta}^{\Delta} dM\sin(Mx_0)\,e^{-\frac{M^2}{2\sigma}}\; e^{-il\sqrt{\tilde{\phi}^2-M}},\; \tilde{\phi}^2 = \phi^2 + M_0$$

For $\Delta^2 \gg \sigma$ the range of the integral can be extended to $[-\infty, \infty]$ with a small error. We also take $x_0 \sim \mathcal{O}(1)$.

- For sufficiently late time,
$$\tilde{\phi} \gg \sigma^{\frac{1}{4}}$$
the wavefunction takes the form,
$$\hat{\Psi} = \frac{\sqrt{2\pi\sigma}}{2il} e^{-il\tilde{\phi}} \left(e^{-\frac{\sigma}{2}(x_0 + \frac{l}{2\tilde{\phi}})^2} - e^{-\frac{\sigma}{2}(-x_0 + \frac{l}{2\tilde{\phi}})^2} \right)$$
- Sharply peaked if
$$\sigma \gg 1$$
- Norm conserved everywhere. Approximation is valid at all times as long as,
$$M_0 \gg \sigma.$$
- Expectation values such as $\langle l \rangle$, $\langle l^2 \rangle$, $\langle \pi_l \rangle$ can be computed. Agrees with behavior expected from classical geometry.
- Infinite number solutions can be obtained by changing various parameters.

Rindler Basis Quantization

The fact that we could only get normalized wavefunctions for $M < 0$ or bounce sectors seems unsatisfactory. To remedy this let us consider a slightly different quantization procedure. Instead of eigenfunctions of $\hat{M}$ we will consider eigenfunctions of $\hat{G}_3$. The resulting wavefunction can be written in terms of Bessel functions with variables, $\xi = l\phi, \theta = \log\left(\frac{\phi}{l}\right)$.

$$\hat{\Psi} = \int dk \left[a(k)e^{ik\theta}J_{-i|k|}(\xi) + b(k)e^{-ik\theta}J_{i|k|}(\xi)\right]$$

The norm is conserved if the wavefunction is independent of ϕ as $l \to 0$. This results in setting $a(k), b(k)$ modes with $k < 0$ to zero. For $l \to \infty$ we assume that the wavefunction dies fast enough. The norm is given by,

$$\langle \hat{\Psi}, \hat{\Psi} \rangle = \frac{2}{\pi} \int_{k>0} dk \sinh(k\pi)(|a(k)|^2 - |b(k)|^2)$$

Notice that no consideration of M has appeared meaning the above holds for any value of M.

Connection Between the Two

Since the wavefunction in both ways of quantization is annihilated by the Hamiltonian constraint it is natural to ask if the two different forms of the wavefunction are related to each other. To do so change Bessels to Hankels and then take $\xi \to \infty$. Then the following relation can be established,

$$\frac{1}{2\pi}\int \hat{\Psi} e^{-ik\theta} d\theta = \frac{1}{2}\frac{e^{-i\xi}}{\sqrt{\xi}} \begin{cases} a(k)c(-k) & ,k > 0 \\ b(|k|)c(|k|) & ,k < 0 \end{cases}$$

where,

$$c(k) = \sqrt{\frac{2}{\pi}} e^{-\frac{k\pi}{2} + i\frac{\pi}{4}}$$

On the other hand the expanding branch wavefunction in M basis in the limit $\xi \to \infty$ takes the form,

$$\hat{\Psi} = \frac{e^{-i\xi}}{\sqrt{\xi}} e^{\frac{\theta}{2}} \int \rho(M) e^{i\frac{M}{2}e^{-\theta}} dM$$

Using the constraint

$$\int \rho(M) dM = 0$$

the following relation can be derived.

$$\frac{1}{2\pi}\int \hat{\Psi} e^{-ik\theta} d\theta = \frac{1}{2\pi}\frac{e^{-i\xi}}{\sqrt{\xi}}\int_{-\infty}^{\infty} d\theta e^{\frac{\theta}{2}} \int \rho(M) e^{i\frac{M}{2}e^{-\theta}} e^{-ik\theta} dM$$
$$= \frac{1}{2\pi}\frac{e^{-i\xi}}{\sqrt{\xi}}\int \rho(M) dM \left(\frac{2}{M}\right)^{ik-\frac{1}{2}} \Gamma\left(ik - \frac{1}{2}\right) e^{(ik-\frac{1}{2})i\frac{\pi}{2}}$$

Comparing both one gets,

$$a(k) = \frac{-ie^{-k\pi}}{\sqrt{2\pi}}\Gamma\left(ik - \frac{1}{2}\right)\int dM\, \rho(M)\, \left(\frac{2}{M}\right)^{ik-\frac{1}{2}}$$

and,

$$b(k) = \frac{-ie^{k\pi}}{\sqrt{2\pi}}\Gamma\left(-ik - \frac{1}{2}\right)\int dM\, \rho(M)\, \left(\frac{2}{M}\right)^{-ik-\frac{1}{2}}$$

Hartle Hawking Wavefunction- A Check

- The WDW equation admits Hankel function as solutions. The solutions take the form,

$$\hat{\Psi} = \int dm\, e^{m\theta}(A_m H_m^1(\xi) + B_m H_m^2(\xi))$$

$$\xi = \sqrt{uv} = \sqrt{l^2 + c_1^m}\sqrt{\phi^2 + c_2^m}, e^{\theta} = \sqrt{\frac{v}{u}} = \frac{\sqrt{\phi^2 + c_2^m}}{\sqrt{l^2 + c_1^m}}$$

- The Hartle Hawking wavefunction in the expanding branch can be evaluated from the path integral to be,

$$\Psi = (\frac{\phi}{l})^{\frac{3}{2}} e^{-il\phi + 2i\pi^2 \frac{\phi}{l}}$$

- Then choosing $m = 2, c_1^2 = -4\pi^2, c_2^2 = 0$ in the exact solution one gets,

$$\Psi \propto \frac{l\phi^2}{(l^2 - 4\pi^2)} H_2^{(2)}(\phi\sqrt{l^2 - 4\pi^2})$$

- Expanding it in the asymptotic limit ($l \to \infty, \phi \to \infty, \frac{l}{\phi} =$ fixed) we find agreement with the no-boundary wavefunction evaluated from the path integral.

- We established numerically that,

$$\int_0^{\infty} \sinh\left(2\pi\sqrt{M}\right) e^{-il\sqrt{\phi^2 - M}} \propto \frac{l\phi^2}{l^2 - 4\pi^2} H_2^{(2)}(\phi\sqrt{l^2 - 4\pi^2})$$

- Thus our procedure for canonical quantization (LHS) agrees with the path integral results (RHS).

- Note that ρ doesn't integrate to zero and hence this has a diverging norm. Finally this too can be verified from path integral. The divergence in the norm, in the path integral language, arises due to the presence of conformal Killing vectors on the sphere.

FROM EXPANDING TO CONTRACTING

A general wavefunction can be expanded in terms of expanding and contracting modes.

$$\hat{\Psi}_+(\xi,\theta) = \int_{k>0} dk \sqrt{\frac{1}{2\pi}} e^{i\frac{\pi}{4}} \frac{e^{-i\xi}}{\sqrt{\xi}} \left(b(k) e^{-ik\theta} e^{-k\frac{\pi}{2}} + a(k) e^{ik\theta} e^{k\frac{\pi}{2}} \right) = \sqrt{\frac{1}{2\pi}} e^{i\frac{\pi}{4}} \frac{e^{-i\xi}}{\sqrt{\xi}} f(\theta)$$
$$\hat{\Psi}_-(\xi,\theta) = \int_{k>0} dk \sqrt{\frac{1}{2\pi}} e^{-i\frac{\pi}{4}} \frac{e^{i\xi}}{\sqrt{\xi}} \left(b(k) e^{-ik\theta} e^{k\frac{\pi}{2}} + a(k) e^{ik\theta} e^{-k\frac{\pi}{2}} \right) = \sqrt{\frac{1}{2\pi}} e^{-i\frac{\pi}{4}} \frac{e^{i\xi}}{\sqrt{\xi}} g(\theta)$$

$\pm$ refer to expanding/contracting respectively. From the above two equations it is clear that

$$f(\theta + i\pi) = i g(\theta)$$

An immediate consequence of the above relation is that the respective coefficients of the expanding and contracting branches are equal. $\tilde{\rho} = \rho$.

One can also obtain contracting branch wavepackets given an expanding branch wavepacket.

Extension to Negative ϕ Branches

Let us now extend the analysis to negative ϕ. In terms of the Rindler modes we can extend the wavefunction past $\phi = 0$. To do that first note

$$J_{-i|k|}(\xi)e^{ik\theta} \to \alpha(k)l^{-2ik}, \quad \phi \to 0, l\text{held fixed}$$

It can be continued to $\phi < 0$ by simply imposing continuity at $\phi = 0$. In the $\phi < 0$ limit we can solve the WDW equation in terms of the variables

$$\tilde{\xi} = -l\phi, e^{\tilde{\theta}} = -\frac{\phi}{l}, \quad \phi < 0.$$

with solutions of the form $J_{\pm i|k|}(\tilde{\xi})e^{\pm ik\theta}$.

The KG norm in the region $\phi < 0$ continues to be of the same form with the integral now being evaluated on a $\phi < 0$ surface. We can again ensure conservation of norm. The wavefunction is then given by

$$\Psi = \int_{k>0} dk[\tilde{a}(k)e^{ik\tilde{\theta}}J_{-i|k|}(\tilde{\xi}) + e^{-ik\tilde{\theta}}\tilde{b}(k)J_{i|k|}(\tilde{\xi})]$$

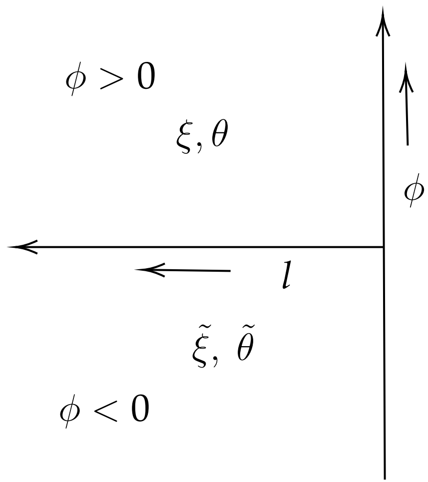


Figure. Extension of the wavefunction to negative ϕ

Now imposing continuity at $\phi = 0$ gives

$$
\begin{aligned}
a(k) &= \tilde{a}(k) \\
b(k) &= \tilde{b}(k)
\end{aligned}
$$

Big Bang/Big Crunch and Black Hole/White Hole Branches

Let us see some consequences of continuing to $\phi < 0$. Let the wavefunction be of the form in the region $\phi < 0$

$$\Psi(l,\phi) = \int dM \hat{\rho}(M) e^{-il\sqrt{\phi^2 - M}} + \int dM \tilde{\hat{\rho}}(M) e^{il\sqrt{\phi^2 - M}}$$

Since the wavefunction from the region $\phi > 0$ can be continued to $\phi < 0$ we learn that

$$\hat{\rho} = \rho, \tilde{\hat{\rho}} = \tilde{\rho}$$

Using the fact that $\tilde{\rho} = \rho$ we end up with just one coefficient function $\rho(M)$.

In the classical solution the big bang/big crunch branch and the black hole/white hole branch are disconnected. It is quite remarkable then to find that in the quantum theory the resulting components in the black hole/white hole branch have the same coefficient functions without any attenuation!

Bounce Sectors

Classically there are two branches in the bounce sector, an expanding branch with $\phi \to \infty$ and a contracting branch with $\phi \to -\infty$. In the wavefunction in the M basis these would correspond to coefficient functions $\rho(M)$ and $\tilde{\hat{\rho}}(M)$.

Recall that the expanding branch corresponds to $\pi_\phi < 0$ and the contracting branch has $\pi_\phi > 0$.

In quantum theory on the other hand we get four solutions. An expanding branch with $\phi > 0$ and $\pi_\phi < 0$ and another expanding branch with $\pi_\phi < 0$ and $\phi < 0$. Similarly in the contracting branch, one branch with $\pi_\phi > 0$ and $\phi < 0$ and another with $\pi_\phi > 0$ and $\phi > 0$.

As in the case of big bang/crunch sector all the coefficient functions turn out to be equal.

General Potential Case

The extension to general potential case in many cases is straightforward. However there are some features in a general potential case that are not present in the JT case.

- **Appearance of Curvature Singularity**: Since $R = U''(\phi)$ it is possible to have genuine Curvature Singularity which is not present in JT. We will see an example of this soon.
- **No Fixed Sign**: R can change sign meaning both dS and AdS can be obtained from the same potential in different ranges of the dilaton.
- **Geodesic Incompleteness**: For potentials of the form ϕ^n with $n > 2$ massive particles reach spatial infinity in a finite time while it takes an infinite affine time for massless particles. We will not delve into this further.
- **Multiple Horizons or Absence of Horizons**: In JT for $M > 0$ we always have two horizons. However for general potential case we can have multiple horizons or even no horizons depending on M.

Considering all the above properties our analysis will necessarily be restrictive in its scope. However we hope to explore more of general potential case in future.

In our analysis we will only consider for the most parts those general potentials which are well behaved with R being of a fixed sign. Additionally our focus will also be on those potentials which asymptote to ϕ^2 in the large ϕ limit.

S Wave Reduction

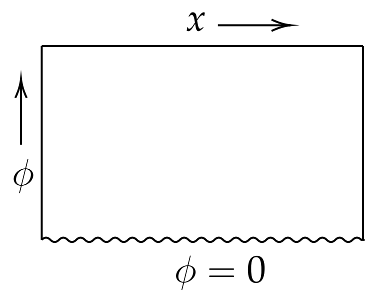
Let us consider an important example of a potential obtained by dimensionally reducing a 4D black hole to 2D assuming spherical symmetry.

$$U(\phi) = 4(\sqrt{\phi^3} - \sqrt{\phi})$$

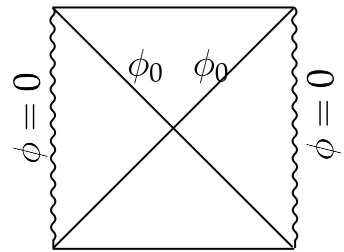
- This potential has a curvature singularity at $\phi = 0$.
- On comparing the 2d metric with the r, t component of the 4D metric one obtains, with μ being the black hole mass,

$$\phi = r^2, M = -4\mu$$

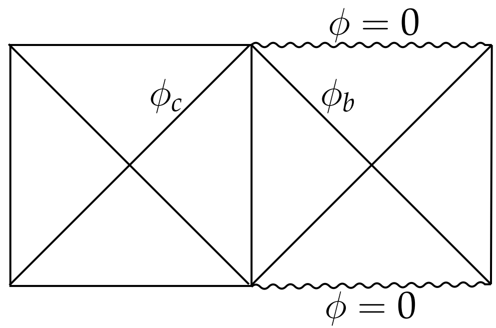
- For $M_0 < M < 0$ one has two horizons. M_0 is called the Nariai limit.
- $M < M_0$ no horizons exist.
- $M > 0$ corresponds to $\mu < 0$ which leads to a timelike singularity. Hence M can not be positive.



$(a)\ M < M_0$



$(b)\ M > 0$



$(c)\ 0 > M > M_0$

An Example of Geodesic Incompleteness

Consider

$$U(\phi) = \phi^n, \quad n > 2.$$

Then we can show that the amount of proper time it takes a massive particle to reach asymptotic infinity is

$$\tau = \frac{2}{n-2} M^{-\frac{1}{2}+\frac{1}{n}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{n}, \frac{3}{2} - \frac{1}{n}, 1 - \frac{p_x^2}{M}\right)$$

which is finite. Here p_x is a conserved quantity. In comparison for a massless particle the affine time is infinite. This suggests that spacetime is geodesically incomplete.

For $n \leq 2$, a similar analysis reveals that the proper time is infinite and thus it's geodesically complete which is expected since the $n = 2$ limit is the dS_2 spacetime.

Spacetimes Where R Changes Sign

An example where R can change sign is

$$R = c_1 + c_2 \tanh(\phi)$$

Here c_1 and c_2 are constants whose signs and relative magnitudes determine Ricci scalar to be positive, negative or zero in suitable limits.

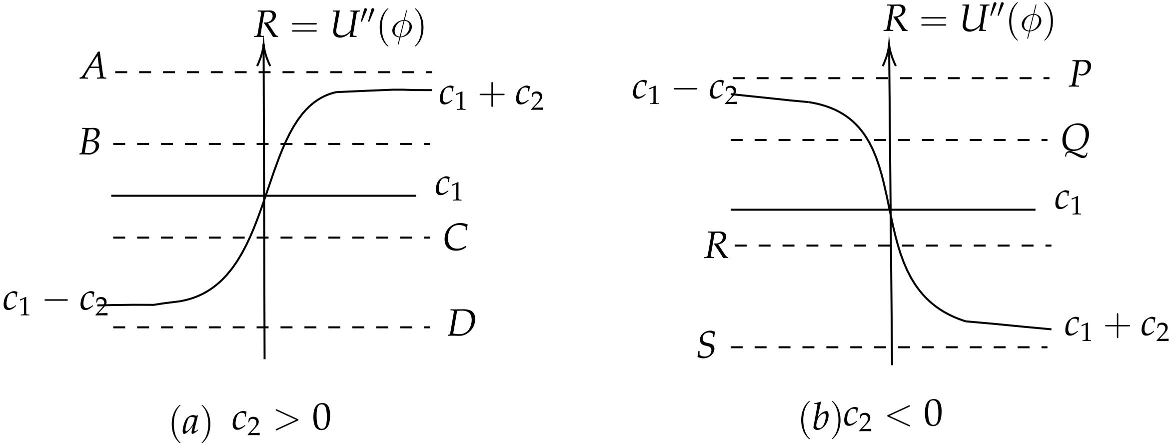


Figure. This plot shows the behaviour of Ricci Scalar R qualitatively.

The potential $U(\phi)$ can be obtained by integrating the Ricci scalar. This introduces two integration constants which correspond to changing $U(\phi)$ by an overall constant and a linear term. The constant term does not enter in the action and is irrelevant. The asymptotic behaviour of the potential is given by

$$
\begin{aligned}
U(\phi) &\rightarrow (c_1 + c_2)\frac{\phi^2}{2} + d\phi, \; \phi \rightarrow \infty \\
U(\phi) &\rightarrow (c_1 - c_2)\frac{\phi^2}{2} + e\phi, \; \phi \rightarrow -\infty
\end{aligned}
$$

We see from the above equation that when $c_1 + c_2 > 0$ and $c_1 - c_2 < 0$ we have a dS region at large positive values of ϕ which becomes AdS space at sufficiently negative values for the dilaton. Alternatively, when $c_1 + c_2 < 0$ and $c_1 - c_2 > 0$, we have a dS bubble inside AdS space.

THERMODYNAMICS

Consider the Euclidean AdS like metric

$$ds^2 = \frac{dr^2}{U(r) - M} + (U(r) - M)dx^2, \ \phi = r$$

where x is periodic with period β. We will require that

$$R = -U''(r) < 0$$

From the usual arguments of smoothness near the horizon where

$$U(r_h) = M$$

we find

$$T = \beta^{-1} = \frac{U'(r_h)}{4\pi}$$

The entropy of the black hole is defined to be,

$$S = 2\pi\phi_h(M)$$

The black hole mass then can be obtained as,

$$dM_{BH} = TdS \implies dM_{BH} = \frac{U'(\phi_h)}{4\pi} 2\pi \phi_h'(M) dM = \frac{dM}{2} \implies M_{BH} = \frac{M}{2}$$

where we used,

$$U(\phi_h) = M \implies U'(\phi_h)\phi_h'(M) = 1$$

The thermodynamic stability of the black hole requires,

$$\frac{dM}{dT} = \frac{dM}{dr_h}\frac{dr_h}{dT} = \frac{16\pi^2 T}{U''(r_h)} > 0$$

Since $U'' > 0, T > 0$ the black holes are thermodynamically stable.
Ref: E. Witten: Deformations of jt gravity and phase transitions, arXiv preprint arXiv:2006.03494, (2020)

Canonical Quantization in General Potential Case

- The canonical quantization in the case of general potential proceeds exactly as in JT.
- The wavefunction now reads,
$$\hat{\Psi} = \frac{1}{l} \exp\left(-il\sqrt{U(\phi) - M}\right)$$
which satisfies the WDW equation,
$$(\partial_l \partial_\phi + \frac{1}{2} l U'(\phi))\hat{\Psi} = 0$$
- The norm is the same as defined in JT. Conservation and finiteness of norm puts stringent conditions on the coefficient functions. The expectation values of various operators can be defined analogously as in JT.
- All of the constraints can be met and one gets an infinite number of states that have finite and conserved norm and are mutually orthogonal.

EXAMPLES

Let us consider the potential

$$U(\phi) = 4(\sqrt{\phi^3} - \sqrt{\phi})$$

with M negative and $0 > M > M_0$. There are two zeroes, r_1, r_2.

In the quantum theory, when $\phi > r_2$ and $U(\phi) > M$, the wave function is oscillatory. The $e^{-il\sqrt{U-M}}$ branch corresponds to future region (F) where the universe expands, while the $e^{il\sqrt{U-M}}$ branch corresponds to the B region where the universe contract (when one goes forward in coordinate time as shown on the Penrose diagram). In contrast when $r_1 < \phi < r_2$ the wave function is exponentially damped or growing.

For $\phi < r_1$ we again get oscillatory branches, growing in white hole region and contracting in black hole regions.

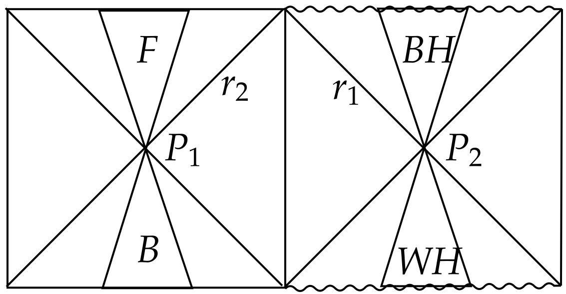


Figure. Penrose Diagram with regions F, B, BH, WH specified.

Let us focus only on those regions where the wavefunction is oscillatory which are F,B,WH,BH. Using the continuity one gets

$$\hat{\rho}(M) + \hat{\tilde{\rho}}(M) = \rho(M) + \tilde{\rho}(M$$

where the hat quantities refer to BH, WH regions.

As discussed earlier, the WKB approximation is only valid when $U, l \to \infty$, this is the case for the F and B regions, when $\phi, l \to \infty$. The spacetime in the WH/BH region is quantum, although the wave function is oscillatory in this region.

Classical Singularities and Their Quantum Resolutions

Let us consider a potential whose second derivative has a delta function divergence.

$$R = U'' \approx C\delta(\phi - \phi_0)$$

The behaviour near this singularity of the potential is

$$U \simeq f(\phi) + C(\phi - \phi_0)\theta(\phi - \phi_0)$$

where f is a smooth function with a finite second derivative at ϕ_0.

The wavefunction will also then be smoothly behaved near ϕ_0, leading to the conclusion that the universe evolves smoothly past the singularity.

This will also be true for a more general wave function which is obtained by taking a linear combination of the solutions above for different values of M.

Exponential Correction Potential

Let us now consider,

$$U'(\phi) = 2\phi + 2\sum_i \epsilon_i e^{-\alpha_i \phi}$$

When $\pi < \alpha_i < 2\pi$, we have a sharp defect and for $\alpha_i < \pi$, it's called a blunt defect.
Refs: E. Witten- Matrix models and deformations of jt gravity, 2020
L. Eberhardt and G. J. Turiaci- 2d dilaton gravity and the weil-petersson volumes with conical defects, 2023

We want to derive the wavefunction corresponding to this potential in path integral formalism asymptotically and then find its exact form. We will for this talk consider sharp defects.

In the $-AdS_2$ case the action is given by

$$S = \frac{1}{2}\left(\int d^2x\sqrt{|g|}(\phi R - 2\phi - 2\sum_i \epsilon_i e^{-\alpha_i\phi}) - 2\int_\partial \sqrt{|\gamma|}\phi K\right)$$

Expanding the path integral to first order first order in ϵ gives

$$\Psi_{HH}(l,\phi) = \Psi_{HH}^{JT}(l,\phi) + \sum_i \epsilon_i \int d^2y\sqrt{|g(y)|}\int DgD\phi e^{-S_{JT}-\alpha_i\phi(y)}$$

In the saddle point approximation we get from the ϕ EOM

$$\sqrt{|g(x)|}(R(x)-2) + 2\alpha_i\delta(x-y) = 0$$

This equation is satisfied by a disk with positive curvature having conical deficit $2\pi-\alpha_i$. Evaluating the path integral one gets the wavefunction to be,

$$\Psi_{HH}^{+}[l_B,\phi_B] = \sqrt{\frac{1}{2\pi}}e^{-il_B\phi_B}\left(\left(\frac{\phi_B}{l_B}\right)^{\frac{3}{2}} e^{\frac{2i\pi^2\phi_B}{l_B}}e^{3i\frac{\pi}{4}} + \sum_i \epsilon_i\frac{\sqrt{\phi_B}}{\sqrt{l_B}}e^{\frac{i(2\pi-\alpha_i)^2\phi_B}{2l_B}}e^{i\frac{\pi}{4}}\right)$$

The exact form of the above wavefunction reads,

$$\Psi_{HH}^{+} = Al_B\left(\frac{v_1}{u_1}H_2^2(\sqrt{v_1u_1}) + \sum_i B_i\sqrt{\frac{v_2}{u_2^i}}H_1^2(\sqrt{v_2u_2^i})\right)$$

$$A = \frac{1}{2}e^{3i\frac{\pi}{2}}, B_i = \epsilon_i, u_1 = l^2 - 4\pi^2, u_2^i = l^2 - (2\pi-\alpha_i)^2, v_1 = U(\phi_B) = v_2$$

- For blunt defects also we found exact wavefunctions that matched with the corresponding results in AdS asymptotically after an appropriate analytic continuation.
- In the sharp defects case when $U'(0) = 0$ the wavefunction has only $\mathcal{O}(\epsilon)$ terms. The wavefunction can be derived from the exact density of states

$$\rho(M) = \frac{\sinh\left(2\pi\sqrt{M}\right)}{4\pi^2} + \sum_i \epsilon_i \frac{1}{2\pi\sqrt{M}} \cosh\left((2\pi - \alpha_i)\sqrt{M}\right)$$

- Using this density of states we then explored the dual matrix model of this theory.
- To do so we calculated first few non-trivial $W_{g,n}$ directly from the density of states assuming the recursion relation holds.
- The resulting answers are then matched with the gravity answers by computing $V_{g,n}$ and doing a Laplace transform.

Hartle Hawking Wavefunction in Various Theories

The HH wavefunction in JT gravity is obtainable from a density of states which for large arguments becomes the exponential of cosmological entropy. Since the large arguments is related to the semi-classical analysis it is interesting to ask if for general potentials also the no boundary wavefunction is related to the exponential of cosmological entropy.

We carry out the no boundary wavefunction for a general potential calculation in the semi-classical regime by following the same path as in the case of JT. For this analysis to work the potential must satisfy a requirement.
Refs: J. Maldacena, G. J. Turiaci and Z. Yang- Two dimensional Nearly de Sitter gravity, (2019)
U. Moitra, S. K. Sake and S. P. Trivedi- Jackiw-Teitelboim Gravity in the Second Order Formalism (2021)

The requirement is

$$U(\phi) = -U(i\phi)$$

The potentials we considered were,

$$U(\phi) = \phi^2(1 + f(\phi^4))$$
$$U(\phi) = (\phi)^{4m+2} \, , m \geq 0$$

This guarantees that,

- For any real and positive β there exists a real solution r_h to the equation,

$$\beta = \frac{4\pi}{U'(r_h)}$$

  This condition comes from the no-boundary condition.
- M defined as $U(r_h) = M$ is positive.
- Ricci scalar doesn't change sign and is always positive.

For the potentials satisfying the above requirement it turns out that,

$$\rho \sim e^{2\pi r_h}$$

where r_h is the cosmological horizon.

The norm for these wavefunctions were found to be divergent.

The potential with exponential correction doesn't satisfy the above requirement and yet we found a no boundary wavefunction. This suggests that there are other methods possible for other potential that is not captured by the above method.

Connection With Matrix Model

Note that in JT the HH wavefunction can be understood as,

$$\Psi^+_{HH}(l,\phi) = e^{-il\phi}\langle \mathrm{Tr}(e^{\frac{ilH}{2\phi}})\rangle = e^{S_0}e^{-il\phi}\int_{E>0} dE\rho_{MM}(E)e^{\frac{ilE}{2\phi}}$$

This begs the question whether one can understand states other than the HH states in this way. While it will be difficult to answer this for arbitrary states one can answer this for a specific type of states.
Consider a deformed theory as in the exponential correction potential. Then it can be shown that the density of states of the deformed theory reads,

$$\rho(M) = f(M)\rho_{MM}(M)$$

Since in path integral one usually computes the wavefunction for $\phi, l \gg 1, \frac{l}{\phi} \gg 1$, if the deformed potential $U(\phi) \to \phi^2$ at large values of ϕ then $\rho(M)$ which gives the HH state in the deformed theory leads to a state different than HH state in JT described by

$$\Psi^+_f(l,\phi) = e^{-il\phi}\langle \mathrm{Tr}(f(H)e^{\frac{ilH}{2\phi}})\rangle$$

Computation of Correlators- A Check of the Proposal

As a check of the proposal let us consider the case of potential with exponential corrections. In this case one has

$$f(M) = 1 + \sum_i 2\pi\epsilon_i \frac{\cosh\left(2\pi - \alpha_i\right)\sqrt{M}}{\sqrt{M}\sinh 2\pi\sqrt{M}}$$

We can then show that

$$\langle \mathrm{Tr}(e^{i\hat{\beta}H} f(H))\rangle = e^{S_0}\left(\frac{e^{\frac{i\pi^2}{\hat{\beta}}}}{4\sqrt{\pi}(-i\hat{\beta})^{3/2}} + \sum_j \epsilon_j \frac{e^{\frac{i(2\pi-\alpha_j)^2}{4(\hat{\beta})}}}{2\sqrt{\pi}(-i\hat{\beta})^{1/2}}\right)$$

This matches with the wavefunction obtained before.

- In contrast when we calculate the same transition amplitudes in the deformed theory for its HH states we get different answers.
- This however is not a contradiction.
- While the deformed theory, obtained by adding the exponential term in the dilaton potential, asymptotically, at late time, does give rise to a physical state in the undeformed JT theory, the two theories are in fact different and the transition amplitudes computed in the two theories need not agree.

A Norm From The Matrix Model

- Let us consider a single realization of SSS matrix model.
- Now consider the matrix, which can be thought of as a Hamiltonian, H, to be an operator acting on a vector space, V. For a single realisation, each eigenvalue of H is associated with a definite state.
- Now let us identify bulk state in the expanding branch, which has eigenvalue M for the mass operator $\hat{M}$, with the eigenstate of H in V with the same eigenvalue M.
- The vector space V is endowed with a standard inner product. A typical state can be written as

$$|f_i\rangle = \sum_i^N f_1(M_i)\ |M_i\rangle, i = 1, 2$$

The inner product then gives the following norm

$$\langle f_i|f_i\rangle = \int dM \rho_{MM}(M)|f_1(M)|^2$$

This inner product will be different from the KG norm.

Finite Rank Completion

At multiple times we mentioned that number of states one gets turns out to be infinity. However the entropy of dS space is finite. Let us try to understand it a little bit more. In the matrix model

$$\rho(E) = \frac{e^{S_{0M}}}{4\pi^2} \sinh\left(2\pi\sqrt{\frac{a^2 - E^2}{2a}}\right)$$

with $-a < E < a$. In the double scaled limit $a \to \infty$ with $\mathcal{E} = E - a$ and $e^{S_{0M}}$ held fixed. Now,

$$L = \int dE \rho_{MM}(E) \sim e^{S_{0M}} \sqrt{a} e^{2\pi\sqrt{a}} \to \infty$$

This explains why the number of states also turn out to be infinity. Understanding the dynamics away from the double scaled limit that is for finite L would correspond to evaluating the path integral for values $\frac{l}{\phi} \sim \mathcal{O}(1)$. This might be a worthwhile task to do to better understand dS entropy.

Path Integral Results and A Different Norm

Before we end let us discuss some path integral results and its connection with Norm.

We look at two such amplitudes. One, which takes a state in the asymptotically far past, in the contracting branch, to a state which is asymptotically, in the far future, in the expanding branch. This should be viewed as the amplitude for a tunneling process, and will be referred to as the tunneling branch.

The second, which takes a state in the expanding branch, asymptotically in the far future, to another state at a later time in the far future (i.e. at larger value of ϕ), also in the expanding branch. This will be referred to at the expanding branch propagator.
CitexxHarlow

The tunneling branch propagator is given by the amplitude

$$\mathcal{K}(l_2,\phi_2;l_1,\phi_1;b) = \exp\left\{\frac{ib^2}{2}\left(\frac{\phi_2}{l_2}-\frac{\phi_1}{l_1}\right) - i(\phi_2 l_2 - \phi_1 l_1)\right\}\left(\frac{\phi_1\phi_2}{l_1 l_2}\right)^{\frac{1}{2}}\frac{1}{8\pi}$$

Integrating over the moduli b, $b^2 \in [0,\infty]$ one gets

$$\mathcal{K}(l_2,\phi_2;l_1,\phi_1) = \frac{1}{2\pi}\exp\{-i(\phi_2 l_2 - \phi_1 l_1)\}\left(\frac{\phi_1\phi_2}{l_1 l_2}\right)^{\frac{1}{2}}\frac{i}{\left(\frac{\phi_2}{l_2}-\frac{\phi_1}{l_1}\right)+i\epsilon}$$

Finally since $\mathcal{K}$ must satisfy WDW equation one can write an exact answer for the propagator in terms of the Hankel functions.

$$\mathcal{K}(l_2,\phi_2;l_1,\phi_1:b) = \frac{1}{16}l_1 l_2\phi_1\phi_2\sqrt{\frac{1}{l_2^2-b^2}}H_1^{(2)}\left(\phi_2\sqrt{l_2^2-b^2}\right)\times$$
$$\sqrt{\frac{1}{l_1^2-b^2}}\left(H_1^{(2)}\left(\phi_1\sqrt{l_1^2-b^2}\right)\right)^*$$

For the Expanding branch propagator one gets

$$\mathcal{K}(l_2, \phi_2; l_1, \phi_1; b) = \exp\left\{ \frac{ib^2}{2} \left(\frac{\phi_2}{l_2} - \frac{\phi_1}{l_1} \right) - i(\phi_2 l_2 - \phi_1 l_1) \right\} \left(\frac{\phi_1 \phi_2}{l_1 l_2} \right)^{\frac{1}{2}} \frac{1}{8\pi} e^{-i\frac{\pi}{2}}$$

Integrating over the moduli

$$\mathcal{K}(l_2, \phi_2; l_1, \phi_1) = 4i \int b db \mathcal{K}(l_2, \phi_2; l_1, \phi_1; b) = 4i \int_{-\infty}^{\infty} d\left(\frac{b^2}{2} \right) \mathcal{K}(l_2, \phi_2; l_1, \phi_1; b)$$

we obtain

$$\mathcal{K}(l_2, \phi_2; l_1, \phi_1) = \exp\{-i(\phi_2 l_2 - \phi_1 l_1)\} \left(\frac{\phi_1 \phi_2}{l_1 l_2} \right)^{\frac{1}{2}} \delta \left(\frac{\phi_2}{l_2} - \frac{\phi_1}{l_1} \right)$$

In this case also we can get an exact form of the propagator.

The expanding branch propagator allows us to calculate the inner product between two states $|l\rangle, |l'\rangle$. Setting $\phi_2 = \phi_1$ we get

$$\langle l|l'\rangle = \mathcal{K}(l, \phi; l', \phi) = l\delta\left(l - l'\right)$$

The resolution of identity following from the above inner product is

$$\int_0^\infty \frac{dl}{l}|l\rangle\langle l| = I$$

Using the above resolution of identity one can check the propagator satisfies the following conditions;

$$\Psi_{HH}(l_2, \phi_2) = \int_0^\infty \frac{dl_1}{l_1}\mathcal{K}(l_2, \phi_2; l_1, \phi_1)\Psi_{HH}(l_1, \phi_1)$$
$$\mathcal{K}(l_2, \phi_2; l_1, \phi_1) = \int_0^\infty \frac{dl}{l}\mathcal{K}(l_2, \phi_1; l, \phi)\mathcal{K}(l, \phi; l_1, \phi_1)$$

Now that we know the resolution of identity we can find the norm given by the path integral which is

$$\langle\Psi|\Psi\rangle = \int \frac{dl}{l} \langle\Psi|l\rangle\langle l|\Psi\rangle = \int \frac{dl}{l} |\Psi|(l,\phi)|^2$$

Let

$$\Psi(l,\phi) = e^{-il\phi} f\left(\frac{l}{\phi}\right), f\left(\frac{l}{\phi}\right) = \int dM \rho(M) e^{\frac{ilM}{2\phi}}$$

The path integral norm then is given by

$$\langle\Psi|\Psi\rangle = \int \frac{dl}{l} \left| f\left(\frac{l}{\phi}\right)\right|^2$$

while the KG norm asymptotically is

$$\langle\Psi|\Psi\rangle = \int \frac{dl}{l}\frac{\phi}{l} \left| f\left(\frac{l}{\phi}\right)\right|^2$$

These are clearly different. However, it is worth noting that both of these expressions are conserved, i.e., independent of the dilaton.

Summary

- Canonically quantized JT gravity and found all gauge invariant states. They are infinite in number
- Chose Dilaton as the physical clock and constructed the space of physical states, by using KG inner product.
- Defined norm, expectation values of gauge invariant operators with KG inner product.
- Explored the constraints imposed by conservation of norm, positivity and finiteness.
- Not all gauge invariant states have conserved norm. Conservation allows only a subset but still infinite in number.
- The well-studied Hartle-Hawking state has diverging norm. Suggests that maybe other gauge-invariant states with finite, conserved norm should be explored more, from the path integral perspective.
- Constructed wave packets that have a good late time classical behavior.

Summary-2

- We found classical solutions corresponding to a general potential $U(\phi)$. We understood some general properties of these spacetimes. We also discussed the classical behavior of potential arising from dimensional reduction.
- The canonical quantization of the general potentials was carried out and the gauge invariant wavefunctions were found.
- We discussed the case of potential with exponential corrections and Hartle hawking wavefunction in various theories.
- In the first case we found the wavefunction corresponding to the potential and explored the dual matrix model.
- In the second case we showed that the no boundary wavefunction can be obtained semi-classically from a density of states whose $\log$ equals the cosmological entropy provided the potential satisfied some requirements.
- Finally the norm of these wavefunctions were divergent.

Open Questions

- We quantized pure JT system. One can add matter into the mix and do a second quantization. This will be an important extension of our work.
- We considered only single boundary sectors. One can consider a third quantization to address the multi boundary case.
- This may also be useful in understanding why dS entropy is finite.
- The canonical quantization procedure we followed is only applicable to those spacetimes where ϕ is timelike. This suggests that the analysis is not complete and more work needs to be done for spacetimes which does not admit a timelike ϕ.
- In HM 2024 the authors quantized the JT system using constant K slices. It will be interesting to see if that story can be repeated in case of a general potential.
- It will also be interesting to work out in detail the no boundary wavefunctions in the path integral method at least in some cases.
- Finally one can also ask whether there is a dual description possible in the case of an arbitrary potential and whether the dual description admits a recursion relation.

References

M. Henneaux- Quantum gravity in two dimensions: Exact solution of the jackiw model, Phys. Rev. Lett. 54 (Mar, 1985) 959–962

C. Crnkovic, E.Witten- Covariant description of canonical formalism in geometrical theories, 1987

P. Saad, S. H. Shenker and D. Stanford- JT gravity as a matrix integral (2019)

J. Maldacena, G. J. Turiaci and Z. Yang- Two dimensional Nearly de Sitter gravity, (2019)

E. Witten: Deformations of jt gravity and phase transitions, arXiv preprint arXiv:2006.03494, (2020)

E. Witten- Matrix models and deformations of jt gravity, 2020

U. Moitra, S. K. Sake and S. P. Trivedi- Jackiw-Teitelboim Gravity in the Second Order Formalism (2021)

L. Eberhardt and G. J. Turiaci- 2d dilaton gravity and the weil-petersson volumes with conical defects, 2023

J.Held, H.Maxfield- The Hilbert space of de Sitter JT: a case study for canonical methods in quantum gravity,2024

More Details On Canonical Quantization

- The canonically conjugate momenta can be evaluated to be,

$$
\begin{aligned}
\pi_N &= 0 = \pi_{N_\perp}, \\
\pi_\phi &= \frac{N_\perp}{2N\sqrt{g_1}} g_1' + \frac{N_\perp' \sqrt{g_1}}{N} - \frac{\dot{g}_1}{2N\sqrt{g_1}} \\
\pi_{g_1} &= -\frac{\dot{\phi}}{2N\sqrt{g_1}} + \frac{N_\perp \phi'}{2N\sqrt{g_1}}.
\end{aligned}
$$

- The constraints take the form,

$$
\begin{aligned}
0 = \mathcal{H} &\equiv \frac{\delta I_{\mathrm{JT}}}{\delta N} = 2\pi_\phi \pi_{g_1} \sqrt{g_1} - \left(\frac{\phi'}{\sqrt{g_1}} \right)' - \sqrt{g_1}\Lambda \\
0 = \mathcal{P} &\equiv \frac{\delta I_{\mathrm{JT}}}{\delta N_\perp} = 2g_1 \pi_{g_1}' + \pi_{g_1} g_1' - \pi_\phi \phi'
\end{aligned}
$$

- The physical wavefunction has to satisfy the constraints above, $\mathcal{H} = 0$ and $\mathcal{P} = 0$, the Hamiltonian and Momentum constraints respectively, i.e

$$\mathcal{H}\Psi = 0$$
$$\mathcal{P}\Psi = 0$$

- The constraints form a first class system.
- There are two issues in solving the constraints.
- The issues of ordering ambiguity and delta function singularity can be dealt with. Delta function singularity arises because of the following action,

$$\sqrt{g_1}\pi_\phi\pi_{g_1}\Psi = \sqrt{g_1}\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta g_1(x)}\Psi[g_1, \phi]$$

Henneaux Formalism

- A way to resolve this by first taking a linear combination of these constraints in the classical theory, to get

$$\pi_{g_1} = \frac{1}{2\sqrt{g_1}}\left(\sqrt{\left(\frac{\phi'}{\sqrt{g_1}}\right)^2 + \phi^2 - M}\right)$$

  where M is the constant derived earlier.
- The other linear combination gives

$$\pi_\phi = \frac{\sqrt{g_1}}{\mathcal{Q}}\left(\left(\frac{\phi'}{\sqrt{g_1}}\right)' + \sqrt{g_1}\phi\right)$$

- The simplified constraint equations then become

$$-i\frac{\delta\Psi}{\delta g_1} = \frac{1}{2\sqrt{g_1}}\left(\sqrt{\left(\frac{\phi'}{\sqrt{g_1}}\right)^2 + \phi^2 - M}\right)\Psi$$

$$-i\frac{\delta\Psi}{\delta\phi} = \frac{\sqrt{g_1}}{\mathcal{Q}}\left(\left(\frac{\phi'}{\sqrt{g_1}}\right)' + \sqrt{g_1}\phi\right)\Psi.$$

- The solution for these equations turns out to be

$$\Psi_\pm[g_1(x), \phi(x)] = e^{\pm i\mathcal{S}[g_1(x),\phi(x)]},$$

$$\mathcal{S}[g_1(x), \phi(x)] = \int dx\left(\mathcal{Q} - \phi'(x)\tanh^{-1}\left(\frac{\mathcal{Q}}{\phi'(x)}\right)\right)$$

$$\mathcal{Q} = \sqrt{(\phi(x)^2 - M)g_1(x) + \phi'(x)^2}$$

Conservation Of Inner Product

- The conservation condition gives,

$$I_N = \partial_\phi \langle \hat{\Psi}_1, \hat{\Psi}_2 \rangle = \frac{i}{2}(\partial_\phi \hat{\Psi}_1^* \hat{\Psi}_2 - \partial_\phi \hat{\Psi}_2 \hat{\Psi}_1^*)|_{l=0}^{l=\infty} + \frac{i}{2}(\partial_\phi \hat{\Psi}_2^* \hat{\Psi}_1 - \partial_\phi \hat{\Psi}_1 \hat{\Psi}_2^*)|_{l=0}^{l=\infty} = 0$$

- Using a compact coefficient function the wavefunction becomes,

$$\hat{\Psi}_i = \frac{1}{l} \int_{-M_{a_i}}^{M_{b_i}} \rho_i(M) e^{-il\sqrt{U(\phi)-M}}$$

  with $M_{a_i}, M_{b_i} > 0$ and $\rho_i(M)$ have M independent phase, $e^{i\theta_i}$. Norm of both wavefunctions is finite and conserved.

- Then it can be shown that the inner product is conserved if

$$\Delta\theta = \theta_2 - \theta_1 = n\pi, n \in \mathcal{Z}$$

- This is an extra constraint on the coefficient functions. And it can be easily met by taking both to be real.

MATCHING OF $W_{g,n}$

- $W_{g,n}$ is given in terms of resolvent as follows,

$$W_{g,n}(z_1, \dots z_n) = (-1)^n 2^n z_1 \dots z_n R_{g,n}(-z_1^2, \dots, -z_n^2)$$

- Given the partition function one can determine the density of states leading to such a partition function.
- Then using that density of states and recursion relations one can then determine any $W_{g,n}$.
- For $g = 0, n = 3$ one gets in the case of JT potential with exponential correction,

$$W_{0,3}(z_1, z_2, z_3) = \frac{2}{z_1^2 z_2^2 z_3^2} \frac{1}{2 - \epsilon_1(4\pi - \alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)}$$

Refs: P. Saad, S. H. Shenker and D. Stanford- JT gravity as a matrix integral (2019)

- From the gravity side one can calculate the volume $V_{g,n}$ and then $W_{g,n}$ is given by,

$$W^{(m)}_{g,n}(z_1,\ldots,z_n) \propto \prod_{i=1}^{n} \int_0^\infty b_i db_i e^{-b_i z_i} V^{(m)}_{g,n}(b_1,\ldots,b_n,\alpha_{i_1},\ldots,\alpha_{i_m})$$

- In the case of JT $V_{g,n}$ are explicitly given. One can obtain $V_{g,n}$ in the case of JT with exponential correction by doing the analytic continuation $b \to i(2\pi - \alpha)$.
- Then using those $V_{g,n}$ one can do the above Laplace transformation to obtain $W_{g,n}$. This we carried out in an ϵ expansion.
- To $\mathcal{O}(\epsilon^2)$ one gets from the gravity side,

$$W_{0,3} = \frac{1}{z_1^2 z_2^2 z_3^2}\left(1 + \frac{\epsilon_1}{2}(4\pi - \alpha_1 - \alpha_2)(\alpha_1 - \alpha_2) + \frac{\epsilon_1^2(\alpha_1-\alpha_2)^2(\alpha_1+\alpha_2-4\pi)^2}{4}\right)$$

- This matches with the earlier answer up to the required order.

Derivation of HH Like State in the case of General Potentials

Consider the $-AdS_2$ like metric given by,

$$ds^2 = -\frac{dr^2}{U(r) - M} - (U(r) - M)\frac{dx^2}{A^2}, \phi = ir$$

The horizon r_h is located at,

$$U(r_h) = M$$

and the usual arguments regarding the smoothness near the horizon leads to,

$$\beta = \frac{4\pi}{U'(r_h)} = \frac{1}{A}$$

The on-shell action in the $-AdS$ case gives

$$-S_{-AdS} = -\frac{i}{A}(U(\phi_B) + M) + \frac{\phi_h}{2A}U'(r_h) = iS_{dS}$$

The dS like metric is given by,

$$ds^2 = -\frac{d\rho^2}{U(\rho)+M} + (U(\rho)+M)\frac{dx^2}{A^2}, \phi = \rho$$

From the above metric we have,

$$\frac{1}{A} = \frac{l_B}{\sqrt{U(\rho_B)+M}}$$

Using the above relation and expanding the action to $\mathcal{O}(M)$ we get

$$\Psi_{HH}(l_B, \phi_B) = \exp\left(-il_B\sqrt{U(\phi_B)} - \frac{il_B M}{2\sqrt{U(\phi_B)}} + 2\pi\phi_h\right)$$

This wavefunction can be derived from the integral

$$I(\beta) = \int \rho(\tilde{M})e^{-\frac{\beta\tilde{M}}{2}}d\tilde{M}, \rho = e^{2\pi\phi_h}$$

by evaluating it in saddle point approximation.

Thank You All For Listening