Combinatorial Optimization 2024 - Problem Set

April 3, 2024

Instructions:

1. Problems marked with - must be solved on your own without consulting anyone or anything. Problems marked with * are compulsory.
2. Problems marked with + are either difficult or need more background than what is covered in class. They are meant to encourage you to read and understand relevant material on your own. So you may refer to a source and understand and write the solution. Of course, if you cannot solve them, you are welcome to approach me.
3. For the unmarked problems, you are allowed, only after thinking on your own, to discuss with someone or to consult a resource. The source i.e. person/reference should be clearly mentioned. It is preferable that, in case you are stuck, you come and discuss your approach with me before resorting to other sources.
4. In any case, you should not copy solutions from others. Whatever you write must be understood and worded by you.
5. **: Important: You need to make one submission every week, make at least 9 submissions in the semester, and submit at least 12 problems in all. There will be 4 marks per submission or 3 marks per problem, and the minimum of the two will be awarded.**
6. I will keep adding problems to this set throughout the duration of the course.

Problems:

1. $(*, +)$ **Basics of LP and polytopes:**
   (a) Consider the system $Ax \leq b$, where A is an $m \times n$ matrix. Thus there are m constraints $a_1^T x \leq b_1, \ldots, a_m^T x \leq b_m$. Consider the set $Z = \{y \mid a_i^T y = b_i$ for $i = 1, \ldots, k,$ and $a_i^T y < b_i$ for $i = k+1, \ldots, m\}$. What is the dimension of Z?
   (b) Let a polytope be given by $P = \{x \mid Ax \leq b, x \geq 0\}$ where x consists of n variables $x_1, \ldots, x_n$ and A is an $m \times n$ matrix, $m \leq n$. Show that the following are equivalent:
      i. A point a in P is an extreme point of P if it cannot be written as a non-trivial convex combination of two distinct points in P.
      ii. A point a is an extreme point of P if n constraints of P are tight at a. A constraint is said to be tight at a if a satisfies the constraint with equality.
      iii. A point a is an extreme point of P if there is a hyperplane $\{x \mid \alpha x = \beta\}$ such that $\alpha a = \beta$ and for any $q \in P, q \neq a$, $\alpha q < \beta\}$.
   (c) Two extreme points q and r of P are said to be *adjacent* if the line segment joining them is an *edge* (i.e. a 1-dimensional face) of P. Show that there is a set of $n-1$ constraints that are tight at both q and r.

2. An edge cover of a graph $G = (V, E)$ is a subset R of E such that every vertex of V is incident to at least one edge in R. Let G be a bipartite graph with no isolated vertex. Show that the cardinality of the minimum edge cover R^* of G is equal to $|V|$ minus the cardinality of the maximum matching M^* of G. Give an efficient algorithm for finding the minimum edge cover of G. Is this true also for non-bipartite graphs?

3. $(-)$ Hall's theorem says that a bipartite graph $G = (A \cup B, E)$ has an A-perfect matching iff $\forall X \subseteq A$, $|N(X) \geq |X|$. Here, an A-perfect matching is one which matches every vertex in A, and $N(x)$ denotes the set of neighbors of vertices in X. Prove Hall's theorem from König's theorem.

4. $(-)$ In this problem, we will generalize Hall's theorem. Let $G = (A \cup B, E)$ be a bipartite graph. For $X \subseteq A$, define deficiency of X to be $def(X) = |X| - |N(X)|$, where $N(X)$ is as above. Define $def(G) = max_{X \subseteq A} def(X)$. Since $def(\emptyset) = 0$, $def(G) \geq 0$.

   (a) Show that the size of a maximum matching in G is $|A| - def(G)$.

   (b) Show that, for any $X, Y \subseteq A$, $def(X \cup Y) + def(X \cap Y) \geq def(X) + def(Y)$.

5. Give an example of a (non-bipartite) graph G, a matching M and a blossom B for M such that a maximum matching M^* in G/B does not extend to a maximum matching in G.

6. Let A be a matrix with entries from $\{0, 1\}$ such that in each row, the 1's are in consecutive positions (i.e. in consecutive columns). Show that such a matrix is TUM.

7. In a bipartite graph, we are interested in matchings of size at most k. Write a set of constraints for this. Is the coefficient matrix TUM? Using this, (or otherwise, if it is not TUM), show that the polytope is integral.

8. Prove that the following three variants of Farka's lemma are equivalent:

   (a) The system $Ax = b$ has a nonnegative solution if and only if every $y \in \mathbb{R}^m$ with $y^T A \geq 0^T$ also satisfies $y^T b \geq 0$.

   (b) The system $Ax \leq b$ has a nonnegative solution if and only if every nonnegative $y \in \mathbb{R}^m$ with $y^T A \geq 0^T$ also satisfies $y^T b \geq 0$.

   (c) The system $Ax \leq b$ has a solution if and only if every nonnegative $y \in \mathbb{R}^m$ with $y^T A = 0^T$ also satisfies $y^T b \geq 0$.

9. Prove that the following three variants of Farka's lemma are equivalent:

   (a) The system $Ax = b$ has a nonnegative solution if and only if every $y \in \mathbb{R}^m$ with $y^T A \geq 0^T$ also satisfies $y^T b \geq 0$.

   (b) The system $Ax \leq b$ has a nonnegative solution if and only if every nonnegative $y \in \mathbb{R}^m$ with $y^T A \geq 0^T$ also satisfies $y^T b \geq 0$.

   (c) The system $Ax \leq b$ has a solution if and only if every nonnegative $y \in \mathbb{R}^m$ with $y^T A = 0^T$ also satisfies $y^T b \geq 0$.

10. Let $T = (V, E)$ be a rooted tree with non-negative edge costs $c : E \rightarrow \mathbb{Z}^+$. Let $(s_1, t_1), ..., (s_k, t_k)$ be a given set of vertex pairs from $V \times V$. A set of edges A in the tree is a multicut for the given pairs if in the forest $T - A$, there is no path from s_i to t_i for any $i \in [k]$. The multicut problem is NP-Hard even in trees. However we can solve a special case and that leads to a 2-approximation in trees. We will focus on the special case.

    - Write an integer programming formulation for the multicut problem in trees with 0/1 variables $x_e, e \in E$.

- Reduce the minimum vertex cover problem in a general graph to the multicut problem in a star. A star on n vertices is a tree with one node as root and all the other nodes are leaves. Use this insight to argue that the LP relaxation is not integral even in stars.
- Suppose we have a special case of multicut in a tree where s_i is an ancestor of t_i in T or the other way around (here we fix a specific root). Prove that the underlying matrix in the LP relaxation is TUM and hence the LP is integral.
- Use the preceding part to argue that the LP relaxation in general trees can be used to obtain a 2-approximation. Hint: Solve the LP and use it to separate s_i or t_i from their least common ancestor by losing a factor of 2 in the LP cost and apply preceding part.

11. You have seen the edge-cover problem in the problem set. Show that the following set of constraints characterize the edge-cover polytope i.e. every edge-cover is a feasible solution of the set of constraints, and also, every vertex of the feasible region of the given constraints is integral:

$$\begin{aligned} \sum_{e \in E[U] \cup \delta(U)} x_e &\geq \frac{|U|+1}{2} \quad \forall U \subseteq V, \quad |U| \text{ odd} \\ 1 \geq x_e &\geq 0 \forall e \in E \end{aligned}$$

Here $E[U]$ denotes the set of edges with both end-points in U. The graph is $G = (V, E)$.

12. Show that $U_{2,4}$ i.e. the uniform matroid over a ground set of size 4, consisting of all ≤ 2 size subsets of the ground set is representable over $\mathbb{F}_3$ but not over $\mathbb{F}_2$.

13. A family F of sets is said to be *laminar* if, for any two sets $A, B \in F$, we have that either (i) $A \subseteq B$, or (ii) $B \subseteq A$ or (iii) $A \cap B = \emptyset$. Suppose that we have a laminar family F of subsets of E and an integer $k(A)$ for every set $A \in F$. Show that (E, I) defines a matroid (a laminar matroid) where:

$$I = \{X \subseteq E : |X \cap A| \leq k(A) \text{ for all } A \in F\}$$

What is the rank function of this matroid?

14. Show that the following is a matroid: $M = (E, I)$ where E is the edge-set of an undirected graph and $I = \{F \subseteq E \mid \text{ every connected component of } F \text{ has at most one cycle}\}$

15. A circuit in a matroid M is an inclusion-wise minimal dependent set, and a cut is an inclusion-wise minimal set which intersects every base. Prove that if a circuit C intersects a cut D, then $|C \cap D| \geq 2$. (Hint: use the strong exchange property, or reasoning similar to its proof.)

16. Given $M = (E, I)$, define the dual matroid $M^* = (E, I^*)$ to have the same ground set; the independent sets I^* are the complements of bases in M, as well as all subsets of these bases. Show that M^* is indeed a matroid.