Linear Matroid Intersection using Determinants.

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Here, we are only interested in intersection of bases of two matroids.
i.e: For two matroids: $M_1 = (E, I_1)$ & $M_2 = (E, I_2)$ on the same ground set E, let the set of Bases in M_1 & M_2 be $\mathcal{B}_1$ & $\mathcal{B}_2$.

Problem 1: Find a $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ or report no such B exists.

Problem 2: Find $|\mathcal{B}_1 \cap \mathcal{B}_2|$, i.e. cardinality of $\mathcal{B}_1 \cap \mathcal{B}_2$

→ Is in Polytime as seen in class.

→ Is very difficult even for linear matroids.

Henceforth we write a matroid as $M = (E, \mathcal{B})$, where $\mathcal{B}$ is the set of bases in M.

This is justified since, I is defined by $\mathcal{B}$ using downward closure & $\mathcal{B}$ is uniquely defined by I by taking max. sets in I.

Defn 1: Let A be a $r \times n$ matrix $(n \geq r)$. & the columns of A indexed by E.

$$B(A) = \{ B \subseteq E \mid |B| = r, \underbrace{\text{columns defined by } B \text{ are linearly independent}}_{\text{i.e. } \det \neq 0} \}$$

$M = (E, B(A))$ is a linear matroid. (if A is of full row rank).

Defn 2: Let A be a $n \times n$ sq. matrix, & S_n the set of permutations on $[n]$.

Then $\displaystyle \operatorname{Det}(A) = \sum_{\sigma \in S_n} (-1)^{\operatorname{sign}(\sigma)} \prod_{i \in [n]} A_{i,\sigma(i)}$: Here sign (σ) is the no. of even cycles in σ, or equivalently, the no. of inversions.

Cauchy-Binet Theorem: Let A_1, A_2 be $n \times m$ & $m \times n$ matrices respectively & $m \geq n$.

Then : $$\operatorname{Det}(\underbrace{A_1 A_2}_{n \times n \text{ matrix.}}) = \sum_{\substack{J \subseteq [m] \\ |J| = n}} \operatorname{Det}(A_1[J]) \operatorname{Det}(A_2[J])$$

submatrix with n rows & n col.s indexed by J

submatrix n rows indexed by J & n col.s.

Let $M_1 = (E, \mathcal{B}_1)$, $M_2 = (E, \mathcal{B}_2)$ be linear matroids where $\mathcal{B}_i := \mathcal{B}(A_i)$ $i \in \{1,2\}$.
Here A_i are $r \times n$ matrices, & $|E| = n$

<u>Lemma 1</u>: Let $J \subseteq E$ & $|J| = r$. Then $J \in \mathcal{B}_1 \cap \mathcal{B}_2$ iff $\det(A_1[J]), \det(A_2[J]) \neq 0$.

<u>Proof</u>: Follows from definition.

Let $D(z) = \begin{bmatrix} z_1 & & & 0 \\ & z_2 & & \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix}$ is an $|E| \times |E|$ diagonal matrix where z_i is a variable representing i^{th} element of E.

Note that $A_1 D(z)$ is just A_1 with z_i multiplied to the i^{th} col. of A_1.

Let $Z_{A_1, A_2}(z) = A_1 . D(z) . A_2^T$ is an $r \times r$ matrix.

Theorem 1: M_1 & M_2 have a common base iff $\det(Z_{A_1,A_2}(z)) \neq 0$.

Proof: $$\det(Z_{A_1,A_2}(z)) = \sum_{\substack{J \subseteq E \\ |J| = r}} \det((A_1 \cdot D(z))[J]) \cdot \det(A_2[J])$$

$$= \sum_{\substack{J \subseteq E \\ |J| = r}} \det(A_1[J]) \cdot \det(A_2[J]) \cdot \prod_{j \in J} z_j$$

$$= \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \det(A_1[B]) \cdot \det(A_2[B]) \cdot \prod_{j \in B} z_j$$

from Lemma 1.

Remark: PIT or Isolation Lemma gives a BPP or RNC algo for finding a common base.

<u>Defn</u>: We say that a matrix pair (A_1, A_2) is pfaffian if $\exists\, c \neq 0$ such that $\forall\, B \in \mathcal{B}(A_1) \cap \mathcal{B}(A_2)$, $\det(A_1[B]) \cdot \det(A_2[B]) = c$.

$\therefore$ If (A_1, A_2) is a pfaffian pair: $\det(Z_{A_1,A_2}(z)) = c \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \prod_{j \in B} z_j$

& $|\mathcal{B}_1 \cap \mathcal{B}_2| = c^{-1} \cdot \det(Z_{A_1,A_2}(\mathbb{1}))$

indicates : $D(z) = I_{n \times n}$

i.e. if (A_1, A_2) is a pfaffian pair, counting common bases of M_1 & M_2 can be done efficiently.

Now we see some applications of the above.

Counting Arborescences:

$G = (V, E, r)$ where $r \in V$, is a directed graph. We want to count # r-rooted arborescences of G.

Let G' be the underlying undirected graph of G.

We know that $F \subseteq E$ is an arborescence if F is a spanning tree in G' & $\forall\ v \in V \setminus \{r\}$, indegree$(v) = 1$ in F.

M_1: $F \subseteq E$ such that F is a spanning tree rooted at r in G'.

M_2: $F \subseteq E$ such that $\forall\ v \in V \setminus \{r\}$, indeg$(v) = 1$.

bases of M_1 & M_2.

A_1: $n \times m$ matrix — rows indexed by V & col. by E.

A_2: " " " " " " " " "

$$A_1(v,e) = \begin{cases} +1 & \text{if } (v,w) = e \\ -1 & \text{if } (u,v) = e \\ 0 & \text{o/w.} \end{cases}$$

$$A_2(v,e) = \begin{cases} 1 & \text{if } (u,v) = e \\ 0 & \text{o/w.} \end{cases}$$

A_1^r, A_2^r be the $n-1 \times m$ matrices obtained by deleting the row corresp. to r in A_1 & A_2.

Claim: A_1^r & A_2^r are linear representations of M_1 & M_2.

<u>proof:</u> A_1^r is linear representation - well known

In A_2^r, each col. has at most one 1.
Let $J \subseteq E$ & $|J| = n-1$. Then $A_2^r[J]$ is non singular iff each col. has exactly one 1, but no row can have two entries to be 1, since then we get two col.s that are the same;
$\therefore$ each row also has exactly one 1, i.e. each vertex other than r has indeg $= 1$.

<u>Claim:</u> If $J \in B_1 \cap B_2$, then $\det(A_1^r[J]) \cdot \det(A_2[J]) = (-1)^{n-1}$
i.e. (A_1, A_2) is a pfaffian pair with constant $= (-1)^{n-1}$.

proof: Perform row & col. operations on $A_1^r[J]$ & $A_2^r[J]$ simultaneously s.t.:
if $u \to v$ is an edge in J, v is above u in row numbering.
columns are $e_1, e_2, \ldots, e_{n-1}$ where e_i is the incoming edge to u_i
where $(u_1, \ldots, u_{n-1})$ is row numbering.

$$\therefore \quad A_1^r[J] = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ \text{something} & & \ddots & -1 \end{bmatrix}, \quad A_2^r[J] = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

& $\because$ the row & col. operations on $A_1^r[J]$ & $A_2^r[J]$ are the same, say k operations in all, then

$$\det(A_1^r[J]) . \det(A_2^r[J]) = (-1)^k . (-1)^k . (-1)^{n-1} . 1 = (-1)^{n-1}.$$

∎.

Therefore we can count #arborescences using $\det(Z_{A_1 A_2}(\mathbb{1})).(-1)^{n-1}$.

Counting Perfect Matchings :

$G = (U \sqcup V, E)$ be a bipartite graph, $|U| = |V| = n$, $|E| = m$.

$M_1 = (E, \mathcal{I}_1)$, $\mathcal{I}_1 = \{F \subseteq E \mid$ each vertex in U incident to at most one edge in $F\}$.

$M_2 = (E, \mathcal{I}_2)$, $\mathcal{I}_2 = \{$ - - - - - - - - V - - - - - - - - - $\}$

clearly $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ are the set of perfect matchings in G.

A_1, A_2 be $n \times m$ matrices.

$$A_1(u,e) = \begin{cases} 1 & u \perp e \\ 0 & o/w \end{cases} \quad \Bigg| \quad A_2(v,e) = \begin{cases} 1 & v \perp e \\ 0 & o/w. \end{cases}$$

Clearly $M_1 = (E, \mathcal{B}(A_1))$ & $M_2 = (E, \mathcal{B}(A_2))$.

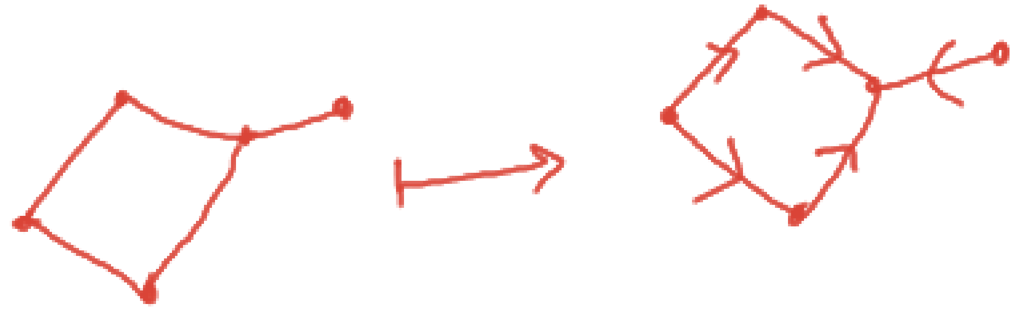
But A_1, A_2 does not form a pfaffian pair. Infact counting perfect matchings in bipartite graphs is #P hard.

$A_1 A_2^T = M$; rows of M are indexed by U & cols by V.

$$M(u,v) = \begin{cases} 1 & (u,v) \in E \\ 0 & \text{o/w.} \end{cases}$$

$$\det(M) = \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} (-1)^{\text{sign}\, \sigma_B} = \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \det(A_1[B]) \det(A_2[B])$$

Now, consider an orientation on E & define $T_{n \times n}$:



$$T(u,v) = \begin{cases} +1 & (u,v) \in E \\ -1 & (v,u) \in E. \\ 0 & \text{o/w.} \end{cases}$$

T defines the orientation

$\vec{A}_1, \vec{A}_2$ be $n \times m$ matrices.

$$\vec{A}_1(u,e) = \begin{cases} 1 & e = (u,v) \\ -1 & e = (v,u). \\ 0 & \text{o/w.} \end{cases}$$

$\vec{A}_2 = A_2$.

$\rightarrow$ clearly $\vec{A}_1, \vec{A}_2$ represent M_1 & M_2.

$\rightarrow$ observe: $T = \vec{A}_1 \cdot \vec{A}_2^{\,T}$

$$\det(T) = \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} (-1)^{\text{sign}\,\sigma_B} \cdot \prod_{e \in B} T_e = \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \det(\vec{A}_1[B]) \det(\vec{A}_2[B]).$$

Call T a pfaffian orientation if for all perfect matchings B, $(-1)^{\text{sign}\,\sigma_B} \prod_{e \in B} T_e$ is a constant (± 1).

$\therefore$ If T is a pfaffian orientation, $\vec{A}_1, \vec{A}_2$ is a pfaffian pair.

$\therefore$ If $\exists$ a pfaffian orientation for G, & we can find it efficiently, then we can count perfect matchings.

Remarks:
- This theory generalizes to all graphs (not just bipartite) using pfaffian instead of determinants.
- Planar graphs admit Pfaffian orientations & can be found in poly time.

Counting Bases in Regular Matroids :

$M = (E, \mathcal{B}(A))$ be a regular matroid & A be the TUM representation.

$\therefore$ $\det(A[B]) \neq 0 \iff B \in \mathcal{B}(A)$.

$\Rightarrow$ $\det(A[B])^2 = 1 \iff B \in \mathcal{B}(A)$.

i.e. $\det(A[B]) \cdot \det(A[B])$ is a constant.

$\therefore$ A, A is pfaffian pair.

$\therefore$ Intersection of M & M gives us # of bases of M.

$$\#\text{bases} = \det(A \cdot A^T) = \sum_{B \in \mathcal{B}(A)} \det(A[B])^2 = \sum_{B \in \mathcal{B}(A)} 1 .$$

- example - counting spanning trees : matrix tree thm.