The Matching Polytope has Exponential Extension Complexity

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Combinatorial Optimisation 2024 Course Presentation
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Extended Formulations

- Let $P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$ be a polyhedron.

- $Q = \{(x,y) \mid Bx + Cy \leq d\} \subseteq \mathbb{R}^n \times \mathbb{R}^h$ and

$$\{x \mid \exists y \text{ s.t } (x,y) \in Q\} = P$$

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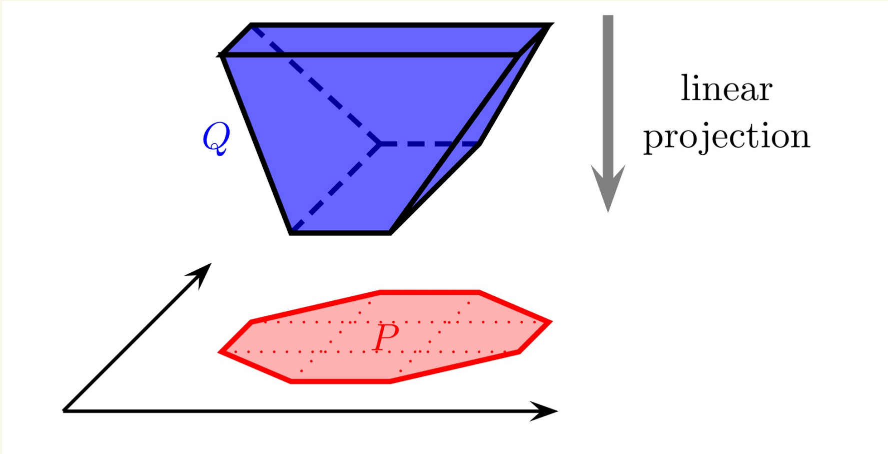
Extension Complexity

$$xc(P) = \min \left(\begin{array}{c} \#\text{ facets of} \\ Q \end{array} \middle| \begin{array}{l} Q \text{ is an} \\ \text{extended form.} \\ \text{of } P \end{array} \right)$$



History I (Compact formulations)

Compact formulations:

- Spanning Tree Polytope [Kipp Martin '91]
- Perfect Matching in planar graphs [Barahona '93]
- Perfect Matching in bounded genus graphs [Gerards '91]
- $O(n \log n)$-size for Permutahedron [Goemans '10] ($\rightarrow$ **tight**)
- $n^{O(1/\varepsilon)}$-size ε-apx for Knapsack Polytope [Bienstock '08]
- ...

Q^n: Is extension complexity always small?

History II (Lower bounds)

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- **Breakthrough:** xc(TSP) $\geq 2^{\Omega(\sqrt{n})}$
  [Fiorini, Massar, Pokutta, Tiwary, de Wolf '12]
- $n^{1/2-\varepsilon}$-apx for clique polytope needs super-poly size
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  Improved to $n^{1-\varepsilon}$ [Braverman, Moitra '13], [Braun, P. '13]

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→ All problems here are NP-hard!

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- $(2-\varepsilon)$-apx LPs for MaxCut have size $n^{\Omega(\log n/\log\log n)}$ [Chan, Lee, Raghavendra, Steurer '13]

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This paper : Perfect matching polytope has $2^{\Omega(n)}$ extension complexity

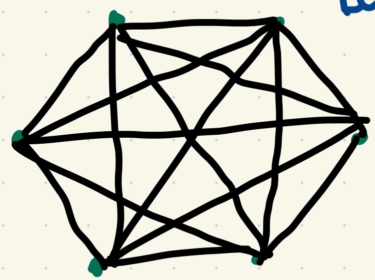
Perfect Matching Polytope

$$x(\delta(v)) = 1 \quad \forall v \in V$$

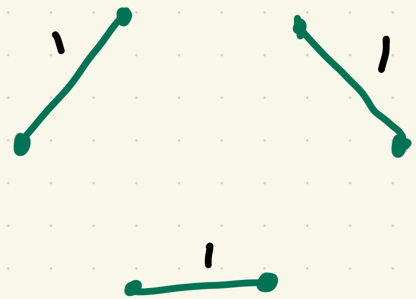
$$x_e \geq 0 \quad \forall e \in E$$

$G = (V, E)$

Let $G = K_n$



Graph



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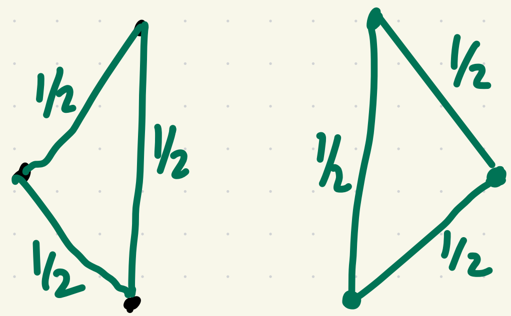

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→ By Edmonds '65

→ Optimization possible in strongly poly time [Edmonds '65]

→ Separation problem polytime

→ $2^{\Theta(n)}$ facets

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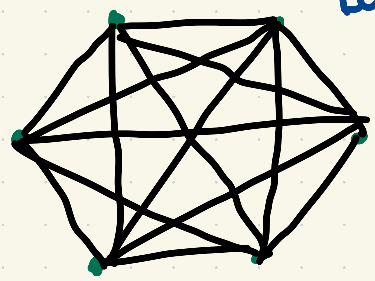

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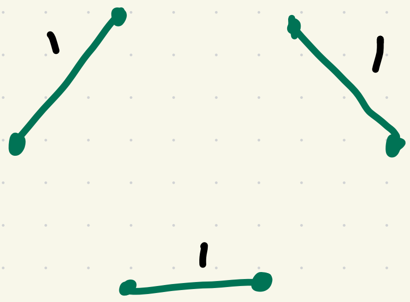

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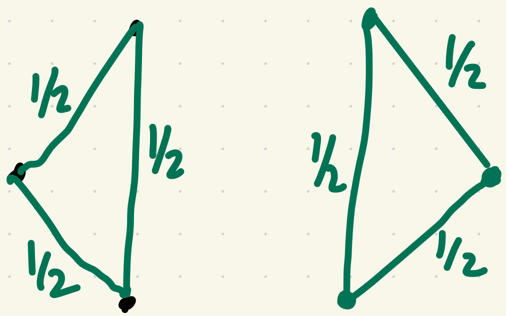

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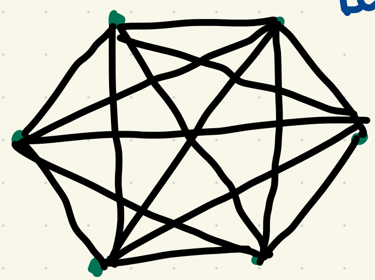

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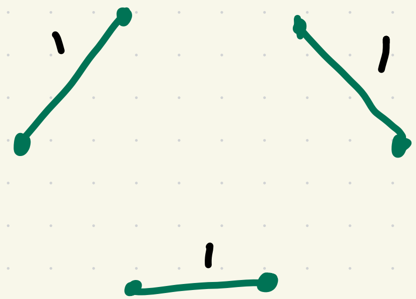


Perfect Matching

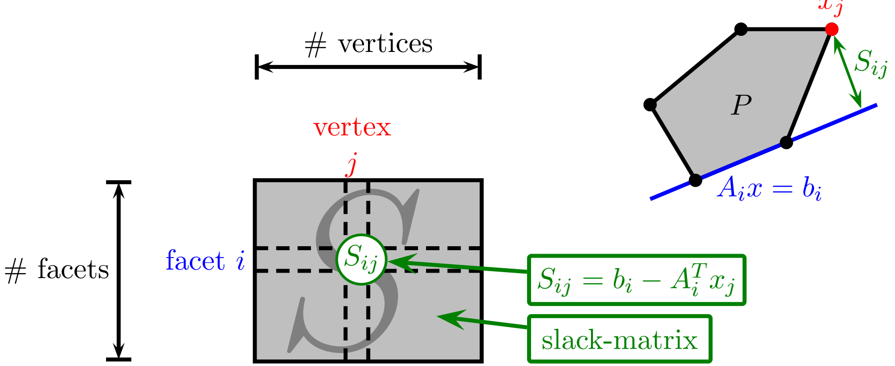
Rothvoss [This paper]:

$$xc\begin{pmatrix}\text{Perfect matching} \\ \text{polytope}\end{pmatrix} \geq 2^{\Omega(n)}$$

Previous: $\geq \Omega(n^2)$

Slack-matrix

Write: $P = \mathrm{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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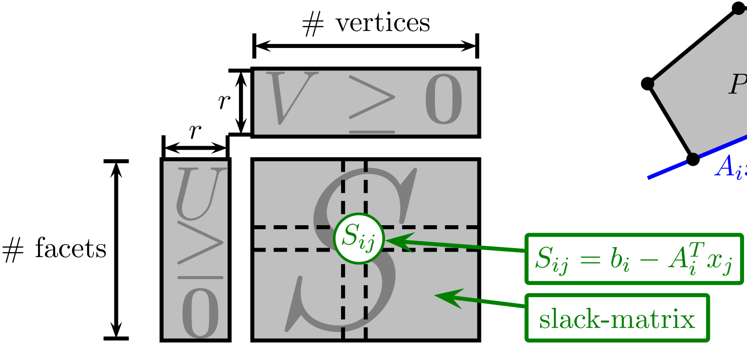

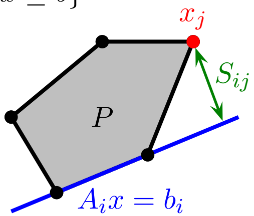

Non-negative rank:

$$\mathrm{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Yannakakis's Theorem ['91]

Let S be the slack matrix of $P = \{x \mid Ax \leq b\}$

then:

$$xc(P) = rk_+(S)$$

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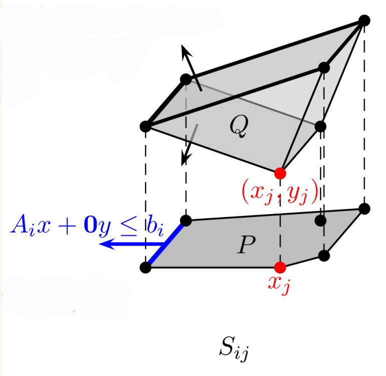
Factorization ⇒ EF

If $S = UV$ s.t $U, V \geq 0$ then

let

$$P = \{x \in \mathbb{R}^n \mid \exists\, y \geq 0 : Ax + Uy = b\}$$

EF ⇒ Factorization



Non negative Rank & Rectangle Covering

$$V = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 4 & 10 & 3 & 5 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 4 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 2 & 0 & 3 \end{bmatrix}$$

$\Rightarrow$ S is sum of 2 rank 1 non negative matrices

Non negative Rank & Rectangle Covering

$\Rightarrow$ $rk_+(S) = \min \left(r \;\middle|\; \begin{array}{l} S \text{ can be written as a sum of } r \\ \text{non-negative rank 1 matrices} \end{array} \right)$

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→ Possible lower bound idea:

Only consider +ve entries as just +ve and 0 entries as 0.

Rectangle Covering Lower bound

$$\begin{array}{cccc} & & V & \\ & & \begin{bmatrix} 0 & 0 & + & + & 0 \\ 0 & + & + & 0 & + \end{bmatrix} & \\ U & \begin{bmatrix} + & + \\ + & + \\ 0 & + \\ 0 & 0 \\ + & 0 \end{bmatrix} & \begin{bmatrix} 0 & + & + & + & + \\ 0 & + & + & + & + \\ 0 & + & + & 0 & + \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & + & 0 \end{bmatrix} & S \end{array}$$

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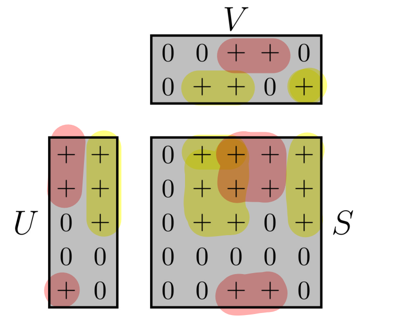

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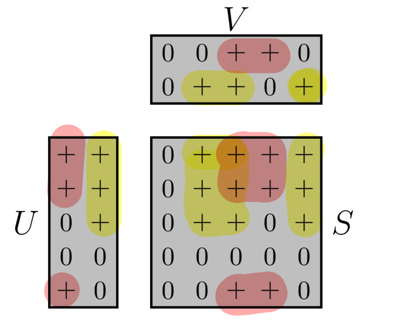

$\Rightarrow rk_+(S) \geq$ rectangle-covering-number(S)

Unfortunately, this bound is horrible
for perfect matching

$\Rightarrow$ Need new techniques!!

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Thank You!

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References :

- Freely used Thomas's amazing presentation from IAS, MSR along with the paper.
- Dr. Yuri Faenza's Strong Relaxations for Discrete Optimization Problems course at EPFL