A short exposition on jump systems

Ankit Gayen

Chennai Mathematical Institute

April 2024

Preliminaries

Definition (*Box*)

Let $V = \{1, 2 \ldots, n\}$. For $x, y \in \mathbb{Z}^V$, define $[x, y] = \{x' \mid min(x_i, y_i) \leq x'_i \leq max(x_i, y_i), \forall i \in V\}$. $[x, y]$ is a **box**

Definition ($(x, y) - step$)

A point $x' \in \mathbb{Z}^V$ is a $(x, y) - step$, if $x' \in [x, y]$ and $d(x, x') = 1$, where $d(x, y) = \Sigma_{i \in V}|x_i - y_i|$

Definition (*Jump system*)

A nonempty set $\mathcal{J} \subseteq \mathbb{Z}^V$ is called a *Jump System* if it obeys the following axiom (any point $x \in \mathcal{J}$ is called *feasible*):
two-step axiom: Given $x, y \in \mathcal{J}$ and a (x, y)-step x', either $x' \in \mathcal{J}$, or there exists a (x', y)-step x" such that x" $\in \mathcal{J}$

Examples

Fig. 1: A jump system and a set that is *not* a jump system

More Examples..

Mention only those examples which you need/can justify.

- **Jump systems in $\mathbb{Z}$**: $\mathcal{J}$ is a jump system in $\mathbb{Z}$ if and only if between any two *feasible* points with distance > 2, there is atleast one *feasible* point
- **Matroids and delta-matroids**: Jump systems contained in the *unit box*($\{0, 1\}^V$) are *delta-matroids*. Among them, those *delta-matroids* with constant coordinate-sum are equivalent to *matroids* and vice versa.(The feasible points are characteristic vectors of *bases*)
- **Degree systems of Graphs**: Let H be a spanning subgraph of a graph G. Define the *degree sequence* of H to be $deg_H \in \mathbb{Z}^V$ such that $deg_H(v)$ equals the degree of v in H. Set of all *degree sequences* of spanning subgraphs of G is called the *degree system* of G. We will see why it's a *jump system* in a moment!

Operations on Jump systems

Mention only the necessary operations. Develop some intuition on board for jump systems before this.

Jump systems are closed under the following (*not exhaustive*):

- **Translation**: Add an integral vector b to every *feasible* point.
- **Reflection**: For some i, replace i^{th} coordinate x_i by $-x_i$ for every *feasible* x.
- **Intersection with a box**: Given a *box* B, $\mathcal{J} \cap B$ is a jump system if it is nonempty.
- **Projection**: Given $S \subseteq V$, replace every feasible point by its restriction to S.
- **Sum**: If $\mathcal{J}_1$ and $\mathcal{J}_2$ are jump systems, then so is $\mathcal{J}_1 + \mathcal{J}_2 = \{x + y \mid x \in \mathcal{J}_1, y \in \mathcal{J}_2\}$
- **Closest points to a box**: Given a *box* B, $\mathcal{J}_B = \{x \in \mathcal{J} \mid d(x, B) = d(\mathcal{J}, B)\}$ is also a jump system.

Now we see why degree systems are jump systems: For a graph G, its *degree system* is the sum of *degree systems* of all its one-edge spanning subgraphs (which are trivially jump systems).

A greedy algorithm for linear optimization in jump systems

- Jump systems acquire one special trait of *matroids* — For an integral vector c, $c^T x$ can be maximized in **polynomial time** over any jump system (assuming it is bounded!).

Greedy

1: Order V as $\{j_1, \ldots, j_n\}$ where $c_{j_1} \geq c_{j_2} \geq \ldots c_{j_k} > 0 = c_{j_{k+1}} = \ldots = c_{j_n}$
2: $\mathcal{J}_0 \leftarrow \mathcal{J}$
3: **for** $i = 1$ to k **do**
4: Set $\alpha = max\{x_{j_i} : x \in \mathcal{J}^{i-1}\}$
5: Set $\mathcal{J}_i = \{x \in \mathcal{J}^{i-1} : x_{j_i} = \alpha\}$
6: **end for**
7: **return** $\mathcal{J}^k$

Do the superscripts and subscripts of J mean different things?

Theorem

Each feasible point $x \in \mathcal{J}^k$ *maximizes* $c^T x$ *over* $\mathcal{J}$

Algorithm Contd.

- But this looks strange! How could it run in polynomial time, if we to calculate *Step 4*(Afterall $\mathcal{J}$ could have size exponential in V!).
- It might be possible that we are given $\mathcal{J}$ as input, but then this algorithm is meaningless! In one scan we find the answer to the optimization problem!
- Not to worry Shioura and Tanaka proved that provided we have access to membership oracle for $\mathcal{J}$, one has the following:

Theorem (*Shioura, Tanaka* '07)

The algorithm Greedy finds an optimal solution in $O(n^2 \log \phi(\mathcal{J}))$ *time, provided a vector in* $\mathcal{J}$ *is given. (where size* $\phi(S)$ *of* S *is* $\phi(S) = \max_{v \in V} \{\max_{y \in S} y(v) - \min_{y \in S} y(v)\}$ *and* $n = |V|$*)* .

- But , we will neither explain the correctness, not the claimed running time here.

Largest restricted factor problem

Definition (2-factor and $\leq$ 2-factor)

A 2-factor(respectively, $\leq$ 2-factor) of a graph $G = (V, E)$ is a set $S \subseteq E$ such that every vertex of G is incident with *exactly* two(respectively, *atmost* two) edges of X

Definition (k-restricted factor)

Is this definition meant for graphs?
Otherwise what does circuit mean?

For a positive integer k, a factor X is *k-restricted* if every *circuit* formed by the edges of X has length *atleast* $k + 1$.

- For a given graph G and integer k, we want to find the largest *k-restricted* factor in G.
- **For k $\geq$ 5**: Hell, Kirkpatrick, Kratchovil and Kriz proved that if the set of circuit lengths to be excluded is not a subset of $\{3, 4\}$, then the problem is $\mathcal{NP}$-hard
- For the *weighted restricted factor* problem, $\mathcal{NP}$-hardness is proved even for bipartite graphs(for $k = 4$).Here, $k = 3$ case remains open,

The slides from here are too dense. Add more explanation, split into multiple slides or prepare to use the board.

Connection with jump systems

- To make the link with jump systems, we ask the question: "*For what values of k, the set $G(k)$ of degree sequences of restricted factors forms a jump system, for any graph G?*". In light of this, we present the following theorem:

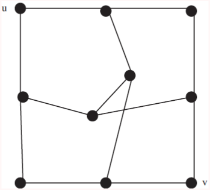
Theorem

For any graph G and any $k \leq 3$, $G(k)$ is a jump system. For any $k > 4$ there exists a graph G such that $G(k)$ is not a jump system.

Proof

For $k <= 2$, restricted factors are same as normal factors, and therefore For any graph G, $G(k)$ is the intersection of its degree system with the box $\{0, 1, 2\}^V$.
For $k = 5$, consider the following graph G.
There are two cycles of length 9, avoiding u and v respectively, in G. Take these two as x and y respectively. Obviously $x, y \in G(5)$ and $x_u = y_v = 0$, $x_w = y_w = 2$, $\forall w \notin \{u, v\}$. Take x' as a (x, y)-step, where $x'_u = 1$ and $x'_w = x_w$ otherwise.

Proof contd.



Now, if $G(5)$ is a jump system, there exists another (x', y)-step $x'' \in G(5)$. Now, either $x''_w = 2, \forall w$, or $x''_u = x''_v = 1, x''_w = 2$ otherwise. For the first case, x'' must be a hamiltonian cycle, which clearly G doesn't have. For the second case, either there is a length 9 path from u to v, or there is a path of length < 4 from u to v. But, none of these holds! Hence $G(5)$ is not a jump system. This graph can be easily modified for $k \geq 6$ by adding more degree 2 vertices.

Connection with jump systems contd.

Proof contd.

Let us now prove $G(3)$ **is** a jump system. Denote $G(3)$ by $\mathcal{J}$.
Let $x, y \in \mathcal{J}$, and let x' be a (x, y)-step. Let u be the component on which x' differs from x(w.l.o.g. assume $x_u < y_u$ then $x'_u = x_u + 1$). Obviously $x' \notin \mathcal{J}$. Therefore, we seek a (x', y)-step x'' such that $x'' \in \mathcal{J}$. In the following to come, we show there is an edge simple path $\mathcal{P}$ such that $X'' = X \triangle \mathcal{P}$ works for x''.

Consider a path from u to some vertex v. We denote the path by $\mathcal{P}_m(u = v_0, v_1, v_2, \dots, v_m = v)$ and denote the path $v_0, \dots, v_i$ by $\mathcal{P}_i$. Also, let $X_i = X \triangle \mathcal{P}_i$. We want $\mathcal{P}$ to satisfy the following properties($\star$):

1. $v_i v_{i+1} \in Y \setminus (X \cup E(P_i))$ for i even
2. $v_i v_{i+1} \in X \setminus (Y \cup E(P_i))$ for i odd
3. X_m is triangle-free

Connection with jump systems contd.

Proof contd.

Our philosophy is simple: start with $\mathcal{P}_0 = u$, which trivially satisfies $\star$, then whenever X_m does not satisfy the requirements for X'', "increase" $\mathcal{P}_m$ to $\mathcal{P}_{m+1}$.

- **When** m **is odd**: If X_m is a (x', y)-step we are done. Else, $deg_{X_m}(v) = x_v + 1 > y_v \implies x_v \geq y_v$. This means $\exists$ an edge $vq \in X \setminus (Y \cup E(\mathcal{P}_m))$. Extend $\mathcal{P}_m$ to $\mathcal{P}_{m+1}$ by setting $v_{m+1} = q$.
- **When** m **is even**: Again, if X_m is already a (x', y)-step, we stop. Else, $deg_{X_m}(v) = x_v - 1 < y_v \implies x_v \leq y_v$. This means $\exists$ an edge $vq \in Y \setminus (X \cup (E(\mathcal{P}_m))$. Now, if $X_m \cup \{vq\}$ is triangle-free, we simply extend to $\mathcal{P}_{m+1}$ by setting $v_{m+1} = q$. Otherwise, X_m contains edges qw, wv for some w forming a triangle.

  Now, if $qw \in X \setminus E(\mathcal{P}_m)$, then extend $\mathcal{P}_m$ to $\mathcal{P}_{m+1}$ by setting $v_{m+1} = q$ and $v_{m+2} = w$. Else, $qw \in E(\mathcal{P}_m) \cap Y$. Now, $wv \notin Y$ (since Y is triangle-free), therefore we must have $wv \in X \setminus (E(\mathcal{P}_m) \cup Y)$. Now $deg_{X_m}(v) = x_v 1 = 1$, whereas $y_v = 2$. Therefore, there exists an edge $v_p \neq v_q$ in $Y \setminus (X \cup E(\mathcal{P}_m)$.

Connection with jump systems contd.

Proof contd.

Suppose that $X_m \cup \{vp\}$ contains a triangle. Then the triangle must have vertices v, p, w. But this would imply that $deg_{X_m}(w) = 3$, a contradiction. Therefore, we can extend $\mathcal{P}_m$ by putting $v_{m+1} = p$.

Since, $\mathcal{P}_m$ is edge simple, we should eventually get the required (x', y)-step!