Topics in Algorithms - Assignment 1

February 5, 2020

1. Construct an instance with 3 men and 3 women that has more than two stable matchings.
2. Find an example such that there exists a (obviously unstable) matching M, in which, some men get strictly better partners than that in the man-optimal stable matching M_o and some men get the same partners as in M_o. This shows that stable matchings are not *pareto optimal*. We have seen in class that they are *weak-pareto optimal*.
3. Construct an example in which the man-optimal stable matching *does not* match any man to his first choice. (Thus the weak-pareto optimality is not obvious.) Is it true that there is always a stable matching that matches some person (man/woman) to his/her first choice? Either prove this or give a counter-example.
4. Consider the following version of the Gale-Shapley algorithm for the stable marriage problem with strict and complete lists: When a woman y accepts a proposal from a man z, the men appearing after z in y's list are deleted from her list and y is deleted from their list. Thus, at the end of the algorithm, we obtain reduced preference lists for all men and women.
   (a) Show that the set of stable matchings is not affected by this i.e. no deleted edge forms a stable pair, nor can it block a stable matching.
   (b) Are all the pairs appearing in the reduced lists stable? If not, give a counter-example.
   (c) If the reduced lists at the end of the man-proposing algorithm are used for executing a woman-proposing algorithm, and are reduced further by applying a similar rule, are all pairs in the resulting lists stable?
5. In the Stable Roommates setting, consider three matchings R_1, R_2, R_3. Let x have partners w, y, z in these matchings such that $w \leq_x y \leq_x z$. Then y is called the *median partner* of x with respect to R_1, R_2, R_3.
   (a) Show that assigning each person his median partner gives a stable matching.
   (b) For a matching R, define $P(R) = R \cup \{(x, y) \mid y \geq_x R(x)\}$. Fix a matching R_0, and consider the sets $P(R_0) \oplus P(R)$ for all matchings R. Use this to construct a semilattice structure that generalizes the dominance relation on Stable Marriages.
6. Consider a roommate matching R. The *regret* of a person x w.r.t. R is the position of $R(x)$ in the preference list of x. The *regret* of R is the maximum regret of any person x w.r.t. R. We shall work towards a polynomial time algorithm to find a *minimum regret stable matching.*
   (a) We extend the concept of regret to a reduced preference table T. Let the least preferred partner of x in T be $l_T(x)$. Then the position of $l_T(x)$ in the original preference list of x is its regret w.r.t. T. As before, the regret of T is the maximum regret of any person x w.r.t. T. Consider an edge of the form $\{x, l_T(x)\}$ such that x has maximum regret w.r.t. T. We call such an edge a *maximum regret edge* w.r.t. T. Show that if a maximum regret edge is not eliminated in the further reduction of T, then we obtain a stable matching that has maximum regret among all stable matchings embedded in T.

(b) In the original roommates algorithm, if multiple rotations are detected in T, they are eliminated in arbitrary order. Replace this arbitrary choice with a criterion that prioritizes the elimination of maximum regret edges from T. Show that your modification does not increase the complexity of the algorithm.

(c) Show that the above modification suffices to find a minimum regret stable matching.