Advanced Algorithms 2023 - Problem Set

November 20, 2023

Instructions:

1. Problems marked with - must be solved on your own without consulting anyone or anything.
2. Problems marked with + are difficult, and you are encouraged to refer to a source and understand and write the solution.
3. You are allowed, only after thinking on your own, to discuss with someone or to consult a resource. The source i.e. person/reference should be clearly mentioned. It is preferable that, in case you are stuck, you come and discuss your approach with me before resorting to other sources.
4. In any case, you should not copy solutions from others. Whatever you write must be understood and worded by you.
5. **: Important: You need to make one submission every week, unless you run out of problems. The submission must be hand-written. Each submission carries 3 marks, capped at 25 total.**
6. Problems are from the exercises of the WS book, unless stated otherwise.

Problems:

1. (-) Show that the 2-approximation for job scheduling problem shown in class can in fact be improved to $2\big(1 - \frac{1}{m}\big)$ where m is the number of machines. Show a similar result for the 4/3-approximation.
2. Show that the nearest neighbor algorithm mentioned in class achieves $O(\log n)$-approximation for metric, symmetric TSP.
3. Problem 2.2
4. Problem 2.3
5. Problem 1.3
6. In case of metric uncapacitated facility location problem, it is not intuitive why there could be a fractional solution with value lower than any integral solution. Construct an instance of the problem to demonstrate this.
7. Problem 1.5(a)
8. Problem 4.3
9. Problem 4.6
10. Problem 5.8
11. (-) Express the following problems as linear programs with relaxed integrality constraints:
    (a) Maximum flow. Construct its dual and show that it represents the minimum s-t cut problem. [1]

[1] There are several ways of writing the max-flow LP. The min-cut interpretation of dual will be easy from one of them.

(b) Shortest s-t path problem in a directed graph, with non-negative weights.
(c) The bin-packing problem.

12. Let (u, v) be an arbitrary edge in a flow network G. Prove that if there is a minimum (s, t)-cut (S, T) such that $u \in S, v \in T$ then there is no minimum $(s - t)$-cut (S', T') such that $u \in T', v \in S'$.

13. Give efficient algorithms to determine whether a given network contains
    (a) a unique max-flow
    (b) a unique minimum (s, t)-cut

14. Give an efficient algorithm to find a minimum (s, t)-cut in G that has the smallest number of edges among all minimum (s, t)-cuts.

15. Observe that any linear combination of (s, t)-flows is itself an (s, t)-flow. This implies that the set of all (not necessarily feasible) (s, t)-flows in any graph actually defines a real vector space, which we can call the *flow space* of the graph.
    (a) Prove that the flow space of any connected graph $G = (V, E)$ has dimension $E?V + 2$.
    (b) Let T be any spanning tree of G. Prove that the following collection of paths and cycles define a basis for the flow space:
        - The unique path in T from s to t;
        - The unique cycle in $T \cup \{e\}$, for every edge $e \notin T$.
    (c) Let T be any spanning tree of G, and let F be the forest obtained by deleting any single edge in T. Prove that the following collection of paths and cycles define a basis for the flow space:
        - The unique path in $F \cup \{e\}$ from s to t,for every edge $e \notin F$ that has one end-point in each component of F;
        - The unique cycle in $F \cup \{e\}$, for every edge $e \notin F$ with both endpoints in the same component of F.
    (d) Prove or disprove the following claim: Every connected flow network has a flow basis that consists entirely of simple paths from s to t.

16. **Gomory-Hu trees:**
    (a) Given a connected (and not necessarily simple) graph $G = (V, E)$ where each vertex has degree at least k (and to avoid triviality, assume that $|V| \geq 3$), prove the following: there exist at least two pairs of vertices (u, v) so that there are k edge-disjoint paths linking u and v. You may want to look at the definition of Gomory-Hu trees very carefully.
    (b) Prove that if the Gomory-Hu tree of G contains all $(n - 1)$ distinct weights, then G can have only one minimum weight cut.

17. **Lovász Local Lemma:**
    (a) Prove the following generalized Local Lemma. Let $A_1, A_2, \ldots, A_n$ be "bad" events in a probability space. Suppose there exists a sequence of numbers $x_1, x_2, \ldots, x_n \in [0, 1)$ such that for all i,

    $$Pr(A_i) \leq x_i \prod_{i \sim j} (1 - x_j) \qquad (1)$$

    Then

    $$Pr(\bigcap_{i=1}^{n} \bar{A}_i) \geq \prod_{i=1}^{n} (1 - x_i) > 0 \qquad (2)$$

    Here $i \sim j$ denotes that A_i depends on A_j.

(b) A hypergraph $H = (V, E)$ is defined as a finite set V of vertices, and E is a collection of subsets of V , called hyperedges. A hypergraph is called k-uniform if $|e| = k$ for all $e \in E$, that is, every edge contains exactly k vertices. In this setting, a typical graph is just a 2-uniform hypergraph. The degree of a vertex in H is the number of edges of which it is a member. A hypergraph is called 2-colorable if we can assign two colors to the vertices in such a way that every edge includes vertices of both colors. That is, no edge is monochromatic. Use the Lovàsz Local Lemma to prove that if H is a k-uniform hypergraph having maximum degree at most $2k?3?1$, then H is 2-colorable.