Advanced Algorithms Assignment 2

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1 Problem 1

Given an undirected graph $G = (V, E)$, $|V| = n$ color its vertices with the minimum number of colors so that the two endpoints of each edge receive distinct colors.

- Give a greedy algorithm for coloring G with $\Delta + 1$ colors, where Δ is the maximum degree of a vertex in G.
- Give a polynomial time algorithm for 2-coloring a bipartite graph
- Give an algorithm for coloring a 3-colorable graph with $O(\sqrt{n})$ colors
- Give an algorithm to color a 4-colorable graph with $O(n^{\frac{2}{3}})$ colors

2 Problem 2

In the **uncapacitated facility location problem**, we have a set of clients D and a set of facilities F. For each client $j \in D$ and facility $i \in F$, there is a cost $c_{i,j}$ of assigning client j to facility i. Furthermore, there is a cost f_i associated with each facility $i \in F$. The goal is to choose a subset of facilities $F' \subseteq F$ so as to minimize the total cost of the facilities in F' and the cost of assigning each client $j \in D$ to the nearest facility in F'. In other words, we wish to find F' so as to minimize

$$\sum_{i \in F'} f_i + \sum_{j \in D} \min_{i \in F'} c_{i,j}$$

- Give a greedy $O(\texttt{log}|T|)$ approximation for the uncapacitated facility location problem.
- By a reduction from set cover or otherwise, show that the uncapacitated facility location problem is as hard to approximate as the set cover problem.

3 Problem 3

Let E be a set of elements, and there are t subsets $S_1, ..., S_t \subseteq E$. The goal is to choose k subsets such that we maximize the size of the covered set.(Basically the number of covered elements)

- Consider a local search algorithm that starts with any solution $S_{i_1}, ..., S_{i_k}$ and tries to make a local improvement by removing any set from the current solution and adding some other set. Show that the locally optimal solution is a 2-approximation
- Now consider a greedy algorithm that iteratively picks the set that maximizes the number of uncovered elements until k sets are chosen. Argue that the solution is always $\frac{e}{e-1}$ approximation

4 Problem 4

In the *hitting set problem*, we are given a ground set E and a collection of sets $S_1, ..., S_m \subseteq E$. Our goal is to choose a collection of elements $F \subseteq E$ such that for any i, $S_i \cap F \neq \phi$, while minimizing the size $|F|$.

- Argue formally that it is the same as set cover. What approximation results follow?
- Now consider a variant of hitting set where each set S_i is said to be satisfied by $F \subseteq E$ if $|F \cap S_i| = 1$. Our goal is to choose $\{F : F \subseteq E$ that maximizes the number of satisfied sets$\}$. Call this the *unique hitting set problem*. Show a constant factor approximation algorithm for unique hitting set when all sets S_i have the same size. (Hint: Randomized algorithm)
- Use the results from the previous question to show a logarithmic factor approximation algorithm for *unique hitting set problem*.
- We say that the instance $(E, \{Si\}_{i=1}^m)$ satisfies the *perfect hitting property* if there is a collection $F \subseteq E$ such that every set is satisfied by F. Given an instance of unique hitting set with perfect hitting property, show an $\frac{e}{e-1}$ approximation algorithm by LP rounding.
- Suppose that all sets have size 2. Do you think this problem is NP-hard, or there is a polynomial time algorithm? What if we know that the sets S_i satisfy both $(\forall i)|S_i| = 2$ and perfect hitting property? Is it polynomial time solvable?.