Assignment 3

October 19, 2018

1. Find out which of the following systems are groups, and if not, why not.
   - Set of all integers with subtraction operation.
   - Set of all real 2×2 matrices with non-zero determinant under matrix multiplication.
   - Set of all rational numbers with odd denominators under addition.
   - The set $1, -1$ under multiplication.
2. List all the elements of S_3, the set of permutations on 3 elements. Label all the permutations in S_3 in terms of the permutations $\pi = (1, 2)$ and $\psi = (1, 2, 3)$.
   - Is this an Abelian group? Justify.
   - Find the subgroups of S_3 and their left and right cosets. Which of the subgroups is(are) normal and why?
   - Give a homomorphism from S_3 to $1, -1$. What is the kernel of this homomorphism?
3. Show that any group of order 3, 4, or 5 is Abelian.
4. Let G be a group of even order. Show that G has an element a such that $o(a) = 2$.
5. Consider a homomorphism $\phi : G \to G'$. Show the following:
   - The kernel of ϕ i.e. $Ker(\phi)$ is a normal subgroup of G.
   - The image of ϕ is a subgroup of G'.
   - The image of ϕ is isomorphic to the quotient group $G/Ker(\phi)$.
   - If ϕ is surjective, then G' is isomorphic to $G/Ker(\phi)$.
6. 1.2.19 from D.B.West
7. If G is a group in which $(a \cdot b)^i = a^i \cdot b^i$ for three consecutive integers i for all $a, b \in G$, show that G is abelian. Also show that, having this property for just two consecutive integers does not imply that G is abelian.
8. (a) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

   where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

   (b) Let H be the subgroup of G above defined by $H = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad - bc = 1\}$. What is $o(H)$?
9. If $a \in G$, define $N(a) = \{x \in G \mid ax = xa\}$. Show that $N(a)$ is a subgroup of G. It is called the *normalizer* or *centralizer* of a in G.

10. If H is of finite index in G prove that there is a subgroup N of G, contained in H, and of finite index in G such that $aNa^{-1} = N$ for all $a \in G$. Can you give an upper bound for the index of this N in G?

11. Let G be an abelian group and let G have elements of orders m and n. Prove that G has an element whose order is the least common multiple of m and n.

12. Let G be a group and A, B subgroups of G. If $x, y \in G$ define $x \sim y$ if $y = axb$ for some $a \in A, b \in B$. Prove

    (a) The relation $\sim$ is an equivalence relation.

    (b) The equivalence class of x is $AxB = \{axb \mid a \in A, b \in B\}$. ($AxB$ is called a *double coset* of A and B in G.)

13. Prove that the two permutations $(1, 2)$ and $(1, 2, \ldots, n)$ generate $\mathcal{S}_n$ which is the group of all permutations on n elements.

14. Let G be the group $\{e, a, b, ab\}$ of order 4, where $a^2 = b^2 = e$ and $ab = ba$. Find the permutations of $\mathcal{S}_4$ corresponding to each element of G *(called the permutation representation of G).*