Chennai Mathematical Institute | Due Date: –/–/–

Problem Set-1

Theoretical Foundations of Computer Science

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Problem1. Let G_n be the graph whose vertices are the permutations of $\{1, 2, \ldots, n\}$, with two permutations $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ adjacent if they differ by interchanging a pair of adjacent entries. Prove that G_n is connected.

Problem2. The **k-dimentional cube** or **hypercube** Q_k is the simple graph whosse vertices are the $k-$tuples with entries in $\{0, 1\}$ and whose edges are pairs of $k-$tuples that differ in exactly one position. Prove the following :
(a) $|V(Q_k)| = 2^k$ and $|E(Q_k)| = k2^{k-1}$.
(b) Q_k is bipartite.

Problem3. Let G be a simple graph on vertices $v_1, v_2, \ldots, v_n$ with m edges. Let $G - v_i$ have m_i edges for $i = 1, 2, \ldots, n$. Prove the following :
(a) $m = \frac{1}{n-2}\Sigma_{i=1}^{n} m_i$
(b) $deg(v_i) = [\frac{1}{n-2}\ \Sigma_{j=1}^{n} m_j\ ] - m_i$ for $i = 1, 2, \ldots, n$.

Problem4. Let G be a graph with no 3-cycles, and let each vertex in G have degree at least k. What is the minimum number of vertices in G ? Can you give an example of such a graph with minimum possible vertices where there is no 3-cycle and degree of each vertex is exactly k ?

Problem5. Let G be a simple graph such that degree of each vertex is at least 3. Prove that G has a cycle of even length. Also prove that G has a cycle with a chord [a chord of a cycle is an edge between non-consecutive vertices along the cycle].

Problem6. Every graph G with average degree d contains a subgraph H such that all vertices of H have degree at least $d/2$ (with respect to H).

Problem7. Prove that every n-vertex graph with $n + 1$ edges contains at least two (possibly overlapping) cycles. Does it always contain at least 3 ?

Problem8. Show that any tree T has at least $\Delta(T)$ leaves, where $\Delta(T)$ is the maximum degree in the graph.

Problem9. Show that a graph is bipartite if and only if every induced cycle has even length. (without using the other characterisation of bipartite graphs, we had seen in the class, and a constructive proof would be appreciated)

Problem10. Prove that every simple graph has a bipartite subgraph with edges $\geq |E|/2$.

Problem11. Let G be a bipartite simple graph with n vertices and e edges. Give a tight lower bound on the number of edges in the complement graph $\overline{G}$.

Problem12. Show that a simple graph with at least two vertices, has at least two vertices that are not cut vertices.

Problem13. A simple graph with n vertices and k components has at most $(n - k)(n - k + 1)/2$ edges. [Hint : use and prove the alebraic identity : $\Sigma_{i=1}^{k} n_i^2 \leq n^2 - (k - 1)(2n - k)$]

Problem14. Show that if graph G has no even cycles then G can have $\leq 3n/2$ edges, where n is the number of vertices in graph G. Is this bound tight ?