Theoretical Foundations of Computer Science (Midsem)

September 25, 2017

1. Let $G = (V, E)$ be a connected, simple, undirected graph (with no self-loops or parallel edges), and each vertex in G has degree exactly 4. Prove or disprove: For any partition of vertices of G into two non-empty sets S and $T = V \setminus S$, there are at least two edges that have one end-point in S and another end-point in T. 8 marks

2. A *Hamiltonian path* in a graph is a simple path that passes through all the vertices. Prove that every tournament has a Hamiltonian path. 10 marks

3. Let $a_1, \ldots, a_n$ be integers. Show that some consecutive sum $a_k + a_{k+1} + \ldots + a_m$ is divisible by n. In other words, there exist $1 \leq k \leq m \leq n$ such that the sum $a_k + a_{k+1} + \ldots + a_m$ is divisible by n. 8 marks

4. Which of the following are posets? Justify your answer. 6 marks
   (a) $(\mathbb{Z}, =)$
   (b) $(\mathbb{Z}, \neq)$
   (c) Set of divisors of a given number $n \in \mathbb{N}$ under divisibility relation.

5. Given n matrices $A_1, \ldots, A_n$, find the number of ways in which the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ can be computed. For example, for $n = 3$, there are two ways: $(A_1 \cdot A_2) \cdot A_3$ and $A_1 \cdot (A_2 \cdot A_3)$. Assume that the matrices have appropriate dimensions i.e. there are integers $p_1, \ldots, p_{n+1}$ such that each A_i has dimension $p_i \times p_{i+1}$. 8 marks

6. Which of the following are groups? Justify your answer. 8 marks
   (a) Set of 2×2 matrices over real numbers under multiplication.
   (b) Set of integers k such that $1 \leq k \leq n$ and $gcd(n, k) = 1$.
   (c) Union of two subgroups of a group
   (d) Intersection of two subgroups of a group

7. **Sterling numbers of second kind:** Find the number of ways of distributing n distinguishable balls into k indistinguishable bins with no bin empty. 12 marks