Theoretical Foundations of Computer Science (Endsem)

November 27, 2017

1. Prove that, for every graph G, either G or $\bar{G}$ is connected. Here $\bar{G}$ is the complement of G, obtained by replacing every edge of G with a non-edge and vice versa. 6 marks

2. Show that every graph with an average degree d has an independent set of size $\frac{n}{2d}$. A set of vertices S is termed as an *independent set* if no two vertices in S have an edge between them. 10 marks

   (*Hint: Form a set S by randomly picking vertices and then eliminate all the edges from S. Using probabilistic method, show the existence of an independent set of given size.*)

3. Let G be a group. Prove that if G has no non-trivial subgroup, then G must be finite of prime order. 10 marks

4. Consider the polynomial ring $R = \mathbb{F}[x_1, \ldots, x_n]$ over a field $\mathbb{F}$. A polynomial in R is said to be *homogeneous of degree* d if all the monomials in the polynomial have total degree d. For example, $3x_1^2x_2 + x_1x_2x_3 + x_2^3$ is a homogeneous polynomial of degree 3.

   Show that the homogeneous polynomials of degree d form a vector space. What is its dimension? 10 marks

5. Let X_1, X_2 be two independent random variables uniformly distributed over a finite field $\mathbb{F}$. Define a set of random variables Y_u, $u \in \mathbb{F}$ as $Y_u = X_1 + uX_2$. Show that $\{Y_u \mid u \in \mathbb{F}\}$ are pairwise independent. 10 marks

6. Let X have possible values $x_1, \ldots, x_n$. Prove that, for a prefix-free binary encoding of these values using lengths $k_1, \ldots, k_n$, it is necessary and sufficient that $\sum_{i=1}^{n} \frac{1}{2^{k_i}} \leq 1$. 8 marks

7. Let G be a simple, triangle-free undirected graph such that each pair of non-adjacent vertices has exactly two common neighbors. Show that G is regular i.e. each vertex of G has the same degree. 8 marks

8. Let G be a graph with only one maximum matching M. Show that M must be a perfect matching. 8 marks