Assignment 2

October 23, 2017

1. 1.2.19 from D.B.West
2. If G is a group in which $(a \cdot b)^i = a^i \cdot b^i$ for three consecutive integers i for all $a, b \in G$, show that G is abelian.
3. (a) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
   where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

   (b) Let H be the subgroup of G above defined by $H = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad - bc = 1\}$. What is $o(H)$?
4. If $a \in G$, define $N(a) = \{x \in G \mid ax = xa\}$. Show that $N(a)$ is a subgroup of G. It is called the *normalizer* or *centralizer* of a in G.
5. If H is of finite index in G prove that there is a subgroup N of G, contained in H, and of finite index in G such that $aNa^{-1} = N$ for all $a \in G$. Can you give an upper bound for the index of this N in G?
6. Let G be an abelian group and let G have elements of orders m and n. Prove that G has an element whose order is the least common multiple of m and n.
7. Let G be a group and A, B subgroups of G. If $x, y \in G$ define $x \sim y$ if $y = axb$ for some $a \in A, b \in B$. Prove

   (a) The relation $\sim$ is an equivalence relation.

   (b) The equivalence class of x is $AxB = \{axb \mid a \in A, b \in B\}$. ($AxB$ is called a *double coset* of A and B in G.)
8. (a) Suppose that α is a characteristic root of the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_p a_{n-p}$ and α has multiplicity 3. Show that α^n, $n\alpha^n$, and $n^2\alpha^n$ are solutions to it.

   (b) Generalize to the case where α is a characteristic root of multiplicity u.
9. A sequence of items $1, 2, \ldots, n$ is waiting to be put into an empty stack. It must be put into the stack in the order given. However, at any time, we may remove an item from the top of the stack. Removed items are never returned to the stack or the sequence awaiting storage. At the end, we remove all items from the stack and achieve a permutation of the labels $1, 2, \ldots, n$. For instance, if $n = 3$, we can first put in 1 then 2 then remove 2, then remove 1, then put in 3, and finally, remove 3, obtaining the permutation $2, 1, 3$. Let q_n be the number of permutations attainable. Find q_n by obtaining a recurrence and solving.

   Additional problems:
10. Prove that the two permutations $(1, 2)$ and $(1, 2, \ldots, n)$ generate $\mathcal{S}_n$ which is the group of all permutations on n elements.

11. Let G be the group $\{e, a, b, ab\}$ of order 4, where $a^2 = b^2 = e$ and $ab = ba$. Find the permutations of $\mathcal{S}_4$ corresponding to each element of G *(called the permutation representation of G)*.

12. Show that a commutative ring D is an integral domain if and only if for $a, b, c \in D$, $a \neq 0$, $ab = ac$ implies $b = c$.

13. Let R be the ring of integers. Show that the set U of all multiples (positive as well as negative) of 17 is an ideal of R. Let V be another ideal of R such that $U \subseteq V \subseteq R$. Then show that either $V = U$ or $V = R$. Can you generalize this? Can the same be said if U is the set of all multiples of 6 instead of 17?

14. If U is an ideal of a ring R and $1 \in U$, then prove that $U = R$. Hence or otherwise, prove that if F is a field, then its only ideals are (0) and F itself.

15. Prove that $x^2 + x + 1$ is irreducible over $\mathbb{F}_2$, the field of integers mod 2.

16. If T is an isomorphism of V onto W, prove that T maps a basis of V onto a basis of W.

17. If $n > m$, prove that there is a homomorphism of $F^{(n)}$ onto $F^{(m)}$ with a kernel W which is isomorphic to $F^{(n-m)}$.