Design and Analysis of Algorithms
Mid-Semester Examination

September 26, 2012

Time: 3 Hours | Marks: 50

Instructions:

1. Solve as many questions as you can.
2. Try to optimize your algorithm as much as possible.
3. Whenever you are asked to design an algorithm, write a correctness proof, and analyze the time complexity.

Questions:

1. You are given a sorted array of n elements which has been *circularly shifted*. For example, $\{35, 42, 5, 12, 23, 26\}$ is a sorted array that has been circularly shifted by 2 positions.

   Give an $O(\log n)$ time algorithm to find the largest element in a circularly shifted array. (The number of positions through which it has been shifted is unknown to you.) **4 marks**

2. A Hamiltonian path in a graph is a simple path that passes through each vertex exactly once. Give an $O(|V| + |E|)$ time algorithm to decide whether there is a Hamiltonian path in a directed acyclic graph.

   4 marks

3. Prove that in the Euclidean TSP case, a minimum length tour does not intersect itself. Hence give an algorithm to find an optimal tour when all the vertices lie on the boundary of a convex polygon (not necessarily a regular one). **6 marks**

4. You are given n currencies $c_1, \ldots, c_n$ and a conversion table R, where $R[i, j]$ denotes the units of c_j that can be bought by one unit of c_i.

   Give an algorithm to determine whether there exist currencies $c_{i_1}, \ldots, c_{i_k}$ such that

   $$R[i_1, i_2] \cdot R[i_2, i_3] \cdot \ldots \cdot R[i_k, i_1] > 1$$

   The condition above asks whether starting from one unit of c_{i_1}, one can go through the conversion sequence $c_{i_2}, \ldots, c_{i_k}, c_{i_1}$ and can finally get more than one units of c_{i_1} back. **6 marks**

5. Given positive integers $r_1, \ldots, r_n$ and $c_1, \ldots, c_n$, design an algorithm to construct a matrix A with 0/1 entries that has the following property:

   For all i, the number of 1s in the ith row is r_i, and number of 1s in the ith column is c_i. If such a matrix does not exist, your algorithm must recognize it. **8 marks**

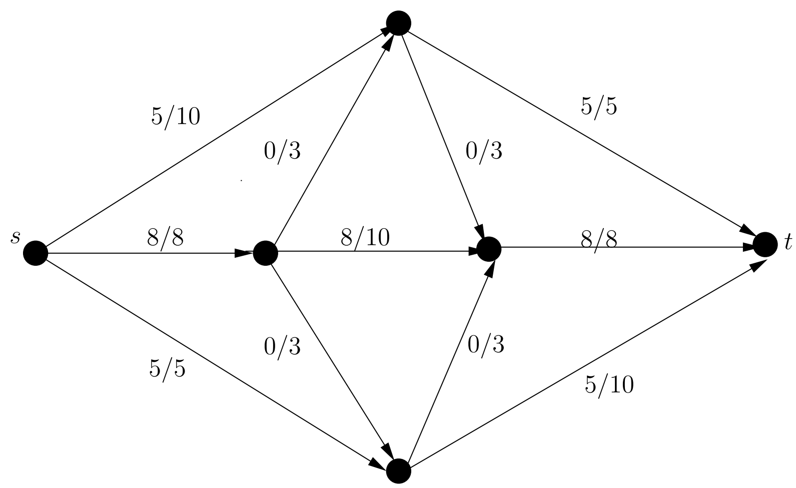
6. Consider the fractional knapsack problem where there are n items $i_1, \ldots, i_n$, with weights $w_1, \ldots, w_n$ and values $p_1, \ldots, p_n$ respectively. You are also given a knapsack which can hold items of total weight at most W. You have to pick up items to put in the knapsack without violating its weight capacity. The goal is to maximize the total value of the items picked. You are allowed to pick fractions of the items, if necessary. **5 marks**

   (a) Prove that this problem has the optimal substructure property.

   (b) Consider the following greedy strategy: Arrange items in non-increasing order of $\frac{v_i}{w_i}$ and pick as many items as possible. If, at some stage, the next item does not fit completely into the knapsack, pick the largest fraction of that item that can fit into the knapsack.

   Prove that this greedy strategy gives an optimal solution.

7. Consider the instance of the network flow problem shown in figure. Every edge has two values of the form x/y, where x is the flow and y is the capacity. (Capacities are shown from the original graph. Thus they are *not* residual capacities.) Answer the following for this instance: **6 marks**



   (a) Compute the residual graph corresponding to the given flow.

   (b) Is the given flow a maximum flow in this network? If not, show an augmenting path in the residual graph.

   (c) What is the minimum capacity of a cut in this instance? Find a cut with the minimum capacity.

8. Several groups of researchers go out for dinner together. To increase new collaboration, they would like to sit at tables such that no two members of the same research group are at the same table. Assuming that there are p groups and q tables, and that the number of people in the ith group is p_i and the capacity of the jth table is c_j, the goal is to accommodate as many researchers as possible. Formulate this problem as a network flow problem. **5 marks**

9. There is a binary counter which counts the number of people entering a building. The counter is incremented by 1 when someone enters the building. (It is *not* decremented when someone leaves the building.) Prove that it takes $O(n)$ time to update the counter from 0 to n, using the accounting method. **6 marks**