Hints/answers and marking scheme to Assignment 3

19/10/2012

1. Split each vertex v into two vertices v_{in} and v_{out}. Join them by a directed edge. A directed edge (u, v) from the original graph now becomes a directed edge (u_{out}, v_{in}). All the edges have capacity 1.

   You need to prove a small claim that the number of vertex-disjoint paths in the original graph is same as the number of edge-disjoint paths in the new graph.

   Correct construction: 2 marks

   Explanation: 2 marks

2. Duals:

   Minimize $10y_1 + 7y_2 + 3y_3$ subject to

$$\begin{aligned} 5y_1 + y_3 &\geq 3 \\ 2y_1 + 3y_2 + y_3 &\geq 4 \\ 4y_2 + y_3 &\geq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

   Maximize $8y_1 + 3y_2$ subject to

$$\begin{aligned} 3y_1 + y_2 &\leq 2 \\ 4y_1 + y_2 &\leq 5 \\ 3y_1 + y_2 &\leq 6 \\ y_1, y_2 &\geq 0 \end{aligned}$$

3. Vertex cover LP for bipartite graph $G = (U, V, E)$:

   Minimize $\sum_{u \in U} x_u + \sum_{v \in V} y_v$ subject to

$$\begin{aligned} x_u + y_v &\geq 1 \qquad \forall (u, v) \in E \\ x_u, y_v &\geq 0 \end{aligned}$$

   Dual:

   Maximize $\sum_{e=(u,v)} p_e$ subject to

$$\begin{aligned} \sum_{e=(u,v)\in E} p_e &\leq 1 \qquad \forall v \in V \\ \sum_{e=(u,v)\in E} p_e &\leq 1 \qquad \forall u \in U \\ p_e &\geq 0 \end{aligned}$$

   Correct LP for vertex cover: 2 marks

   Correct dual: 2 marks

   Interpretation of dual, and application of strong duality: 2 marks

4. (a) $O(n^2)$ algorithm: For each i, j, check if $i, j, i + j$ form a well-spaced triple of 1s.
   Correct algorithm: 2 marks

(b) Interpretation: $i = 2k$, P_i is odd, $P_i \geq 3 \Leftrightarrow \exists$ a well-spaced triple centered at k.

Moreover, P_i is odd $\Rightarrow a_k = 1$, and $\frac{P_i - 1}{2}$ is the number of well-spaced triples centered at k.

If $i = 2k+1$, $\frac{P_i}{2}$ = number of pairs of 1s separated by an even number of positions (i.e. pairs of 1s at positions p, q such that $p + q = 2k + 1$). These are precisely those pairs which can't participate in any well-spaced triple.

Existential interpretation for even i: 1 mark

Counting the number of well-spaced triples looking at the value of P_i: 1 mark

Interpretation for odd i: 1 mark

(c) Compute P in $O(n \log n)$ time. Go over all the P_is where i is even. Count as mentioned above.

Correct counting: 2 marks

(d) Find an even i such that $P_i \geq 3, P_i$ is odd. Go over each of the positions $p, p + i$ in S and check for 1s.

Correctly finding a triple: 1 mark.