# Decision Procedures
## An Algorithmic Point of View

### Gaussian Elimination and Simplex

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# Gaussian's elimination

- Given a linear system Ax = b

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}} \overset{=}{=} \overset{b}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}}$$

- Manipulate A|b to an upper-triangular form

$$\left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \dots & a'_{1k} & b'_1 \\ 0 & a'_{22} & \dots & a'_{2k} & b'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a'_{kk} & b'_k \end{array}\right]$$

# Gaussian' s elimination

- Then, solve backwards from the $k$' s row according to:

$$x_i = \frac{1}{a'_{ii}}(b'_i - \sum_{j=i+1}^{k} a'_{ij}x_j)$$

# Gaussian elimination - example

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 4 \\ 4 & -1 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ -2 & 3 & 4 & 3 \\ 4 & -1 & -8 & 9 \end{array}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ -2 & 3 & 4 & 3 \\ 0 & -9 & -12 & -15 \end{array}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad R3+ = -4R1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 7 & 6 & 15 \\ 0 & -9 & -12 & -15 \end{array}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad R2+ = 2R1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 7 & 6 & 15 \\ 0 & 0 & -\frac{30}{7} & \frac{30}{7} \end{array}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad R3+ = \frac{9}{7}R2$$

And now… $x_3$ = -1, $x_2$ = 3, $x_1$ = 1 problem solved.

# Feasibility with Simplex

- Simplex was originally designed for solving the optimization problem:

$$\max \vec{c}\,\vec{x}$$

s.t.

$$A\vec{x} \leq \vec{b}, \quad \vec{x} \geq \vec{0}$$

- We are only interested in the feasibility problem.

*Is this system feasible ?*

*Is this system optimal ?*

# General simplex

- We will learn a variant called general simplex.
- Very suitable for solving the feasibility problem fast.
- The input: $A\vec{x} \leq \vec{b}$
  - A is a $m \times n$ coefficient matrix
  - The problem variables: $\vec{x} = x_1, \ldots, x_n$
- First step: convert the input to general form

# General form

- General form: $A\vec{x} = 0$ and $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$

- A combination of:
  - Linear equalities of the form $\sum_i a_i x_i = 0$
  - Lower and upper bounds on variables.

# Converting to General Form

- A: Replace $\sum_i a_i x_i \bowtie b_j$ (where $\bowtie \in \{=, \leq, \geq\}$)

  with $\sum_i a_i x_i - s_j = 0$

  and $s_j \bowtie b_j$

- $s_1, \ldots, s_m$ are called the additional variables.

# Example 1

- Convert $x + y \geq 2$

  to: $x + y - s_1 = 0$
  $s_1 \geq 2$

It is common to keep the conjunctions implicit

# Example 2

- Convert

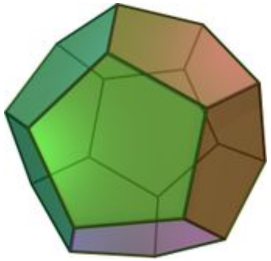
$$\begin{array}{rrl} x & +y & \geq 2 \\ 2x & -y & \geq 0 \\ -x & +2y & \geq 1 \end{array}$$

to:

$$\begin{array}{rrrl} x & +y & -s_1 & = 0 \\ 2x & -y & -s_2 & = 0 \\ -x & +2y & -s_3 & = 0 \\ & & s_1 & \geq 2 \\ & & s_2 & \geq 0 \\ & & s_3 & \geq 1 \end{array}$$

# Simplex basics…

- Linear inequality constraints, geometrically, define a convex polyhedron.



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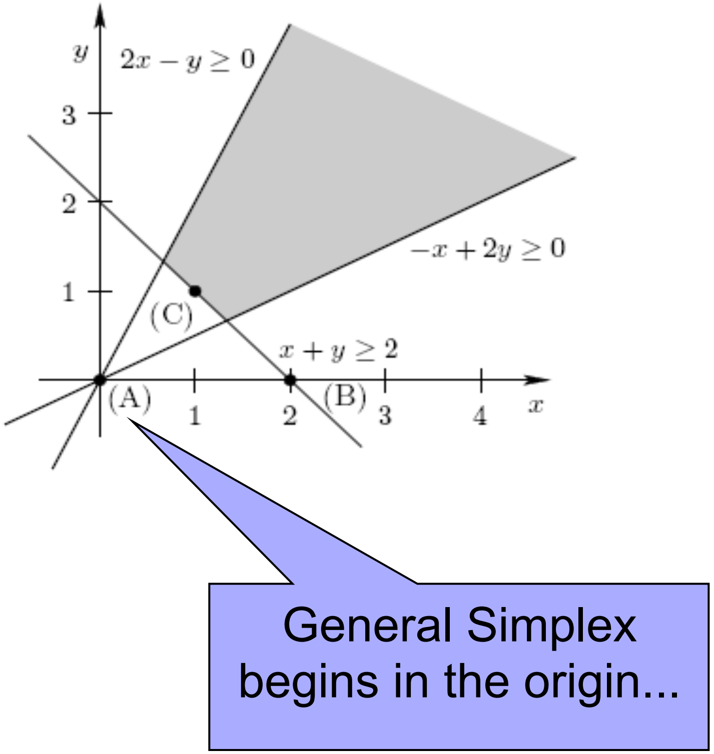
# Our example from before, geometrically

$$\begin{array}{rrl} x & +y & \geq 2 \\ 2x & -y & \geq 0 \\ -x & +2y & \geq 1 \end{array}$$



# Matrix form

- Recall the general form: $A\vec{x} = 0$ and $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$
- Due to the additional variables:
  - now A is an $m \times (n+m)$ matrix.

$$
\begin{array}{rrrl}
x & +y & -s_1 & = 0 \\
2x & -y & -s_2 & = 0 \\
-x & +2y & -s_3 & = 0 \\
 & & s_1 & \geq 2 \\
 & & s_2 & \geq 0 \\
 & & s_3 & \geq 1
\end{array}
$$

$$
\begin{array}{ccccc}
x & y & s_1 & s_2 & s_3
\end{array}
$$

$$
\begin{pmatrix}
1 & 1 & -1 & 0 & 0 \\
2 & -1 & 0 & -1 & 0 \\
-1 & 2 & 0 & 0 & -1
\end{pmatrix}
$$

# The tableau

- The diagonal part is inherent to the general form

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & s_3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

- We can instead write:

$$\begin{array}{c|cc} & x & y \\ \hline s_1 & 1 & 1 \\ s_2 & 2 & -1 \\ s_3 & -1 & 2 \end{array}$$

This is called the tableau

# The tableau

- The tableau changes throughout the algorithm, but maintains its $m \times n$ structure

Nonbasic variables

Basic variables

$$\begin{array}{c|cc} & x & y \\ \hline s_1 & 1 & 1 \\ s_2 & 2 & -1 \\ s_3 & -1 & 2 \end{array}$$

- Distinguish between basic and nonbasic variables
- Initially, basic variables = the additional variables.

# The tableau

- Denote by
  - $\mathcal{B}$ – Basic variables
  - $\mathcal{N}$ – Nonbasic variables
- The tableau is simply a rewrite of the system:

$$\bigwedge_{x_i \in \mathcal{B}} \left( x_i = \sum_{x_j \in \mathcal{N}} a_{ij} x_j \right)$$

- The basic variables are also called the dependent variables.

# The general simplex algorithm

- Simplex maintains:
  - The tableau,
  - an assignment $\alpha$ to all variables
  - The bounds

- Initially,
  - $\mathcal{B}$ = additional variables
  - $\mathcal{N}$ = problem variables
  - $\alpha(x_i) = 0$ for $i \in \{1,\ldots,n+m\}$

# Invariants

- Two invariants are maintained throughout:

1. $A\vec{x} = 0$
2. All nonbasic variables satisfy their bounds

- Can you see why these invariants are maintained initially ?
- We should check that they are indeed maintained

# The general simplex algorithm

- The initial assignment satisfies $A\vec{x} = 0$
- If the bounds of all basic variables are satisfied by $\alpha$, return `Satisfiable'.

- Otherwise... pivot.

# Pivoting

- Find a basic variable $x_i$ that violates its bounds.
  - Suppose that $\alpha(x_i) < l_i$
- Find a nonbasic variable $x_j$ such that
  - $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
  - $a_{ij} < 0$ and $\alpha(x_j) > l_j$
- Why ?

# Pivoting

- Find a basic variable $x_i$ that violates its bounds.
  - Suppose that $\alpha(x_i) < l_i$
- Find a nonbasic variable $x_j$ such that
  - $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
  - $a_{ij} < 0$ and $\alpha(x_j) > l_j$
- Such a variable $x_j$ is called suitable.
- If there is no suitable variable – return 'Unsatisfiable'
  - Why ?

# Pivoting $x_i$ with $x_j$

- Solve equation $i$ for $x_j$:

  From: $$x_i = a_{ij}x_j + \sum_{k \neq j} a_{ik}x_k$$

  To: $$x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

- Swap $x_i$ and $x_j$, and update the $i$-th row accordingly.

  From

| | | | | |
|---|---|---|---|---|
| $a_{i1}$ | … | $a_{ij}$ | … | $a_{in}$ |

  To:

| | | | | |
|---|---|---|---|---|
| $\frac{-a_{i1}}{a_{ij}}$ | … | $\frac{1}{a_{ij}}$ | … | $\frac{-a_{in}}{a_{ij}}$ |

# Pivoting $x_i$ with $x_j$

- Update all other rows:
  - Replace $x_j$ with its equivalent obtained from row $i$:

$$x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

# Pivoting

- Update $\alpha$ as follows:
- Increase $\alpha(x_j)$ by $\theta = \frac{l_i - \alpha(x_i)}{a_{ij}}$
  - Now $x_j$ is a basic variable: it can violate its bounds.
- Update $\alpha(x_i)$ accordingly
  - Q: What is now $\alpha(x_i)$ ?
- Update α for all other basic (dependent) variables.

# Example

- Recall the tableau and constraints in our example:

| | $x$ | $y$ |
|---|---|---|
| $s_1$ | $1$ | $1$ |
| $s_2$ | $2$ | $-1$ |
| $s_3$ | $-1$ | $2$ |

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Initially $\alpha$ assigns $0$ to all variables
- Bounds of $s_1$ and $s_3$ are violated

# Example

- Recall the tableau and constraints in our example:

| | $x$ | $y$ |
|---|---|---|
| $s_1$ | 1 | 1 |
| $s_2$ | 2 | −1 |
| $s_3$ | −1 | 2 |

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- We will solve $s_1$
- $x$ is a suitable nonbasic variable for pivoting
  - It has no upper bound
- So now we pivot $s_1$ with $x$

# Example

- Recall the tableau and constraints in our example:

| | $x$ | $y$ |
|---|---|---|
| $s_1$ | $1$ | $1$ |
| $s_2$ | $2$ | $-1$ |
| $s_3$ | $-1$ | $2$ |

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- Solve 1st row for $x$: $s_1 = x + y \iff x = s_1 - y$
- Replace $x$ with $s_1$ in other rows:

$$s_2 = 2(s_1 - y) - y \iff s_2 = 2s_1 - 3y$$

$$s_3 = -(s_1 - y) + 2y \iff s_3 = -s_1 + 3y$$

# Example

- The new state:

| | $s_1$ | $y$ |
|---|---|---|
| $x$ | 1 | $-1$ |
| $s_2$ | 2 | $-3$ |
| $s_3$ | $-1$ | 3 |

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- Solve 1st row for $x$: $s_1 = x + y \iff x = s_1 - y$
- Replace $x$ with $s_1$ in other rows:

$$s_2 = 2(s_1 - y) - y \iff s_2 = 2s_1 - 3y$$

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# Example

- The new state:

| | $s_1$ | $y$ |
|---|---|---|
| $x$ | 1 | −1 |
| $s_2$ | 2 | −3 |
| $s_3$ | −1 | 3 |

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- What about the assignment ?
- We should increase $x$ by $\theta = \frac{2-0}{1} = 2$
  - Hence, $\alpha(x) = 0 + 2 = 2$
  - Now $s_1$ is equal to its lower bound: $\alpha(s_1) = 2$
  - Update all the others

# Example

- The new state:

| | $s_1$ | $y$ |
|---|---|---|
| $x$ | 1 | $-1$ |
| $s_2$ | 2 | $-3$ |
| $s_3$ | $-1$ | 3 |

$$\begin{aligned} \alpha(x) &= 2 \\ \alpha(y) &= 0 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= 4 \\ \alpha(s_3) &= -2 \end{aligned}$$

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- Now $s_3$ violates its lower bound
- Which nonbasic variable is suitable for pivoting ?
  - That's right… $y$

# Example

- The new state:

| | $s_1$ | $y$ |
|---|---|---|
| $x$ | 1 | −1 |
| $s_2$ | 2 | −3 |
| $s_3$ | −1 | 3 |

$$\begin{aligned} \alpha(x) &= 2 \\ \alpha(y) &= 0 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= 4 \\ \alpha(s_3) &= -2 \end{aligned}$$

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- We should increase $y$ by $\theta = \frac{1-(-2)}{3} = 1$

# Example

- The final state:

| | $s_1$ | $s_3$ |
|---|---|---|
| $x$ | $2/3$ | $-1/3$ |
| $s_2$ | $1$ | $-1$ |
| $y$ | $1/3$ | $1/3$ |

$$\begin{aligned} \alpha(x) &= 1 \\ \alpha(y) &= 1 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= 1 \\ \alpha(s_3) &= 1 \end{aligned}$$

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- All constraints are now satisfied

# Observations

- The additional variables:
  - Only additional variables have bounds.
  - These bounds are permanent.
  - Additional variables exit the base only on extreme points (their lower or upper bounds).
  - When entering the base, they shift towards the other bound and possibly cross it (violate it).

# Observations

- Can it be that we pivot($x_i$,$x_j$) and then pivot($x_j$,$x_i$) and enter a (local) cycle ?
  - No.
  - For example, suppose that $a_{ij} > 0$.
  - We increased $\alpha(x_j)$ so now $\alpha(x_i) = l_i$.
  - After pivoting, possibly $\alpha(x_j) > u_j$
  - But $a_{ij}' = 1 / a_{ij} > 0$, hence $x_i$ is not suitable.

# Observations

- Is termination guaranteed ?
- Not obvious.
  - Perhaps there are bigger cycles.
- In order to avoid circles, we use Bland's rule:
  - determine a total order on the variables.
  - Choose the first basic variable that violates its bounds, and first nonbasic suitable variable for pivoting.
  - It can be proven that this guarantees that no base is repeated, which implies termination.

# General-Simplex with Bland's rule

1. Transform the system into the general form

$$A\vec{x} = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i \ .$$

2. Set $\mathcal{B}$ to be the set of additional variables $s_1, \ldots, s_m$.
3. Construct the tableau for $A$.
4. Determine a fixed order on the variables.
5. If there is no basic variable that violates its bounds, return "Satisfiable". Otherwise, let $x_i$ be the first basic variable in the order that violates its bounds.
6. Search for the first suitable nonbasic variable $x_j$ in the order for pivoting with $x_i$. If there is no such variable, return "Unsatisfiable".
7. Perform the pivot operation on $x_i$ and $x_j$.
8. Go to step 5.