

COMBINATORICS OF THE GENERAL LINEAR GROUP

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ABSTRACT. This is a brief description of material studied and work done by Vipul Naik from the Chennai Mathematical Institute as part of a Summer Exchange Programme with the Ecole Normale Supérieure, in the months of May and June 2007, under the guidance of Professor Olivier Schiffmann. In the first few sections, some ideas are discussed in detail, while the last section is a quick overview of other related ideas.

1. PARALLEL BETWEEN GENERAL LINEAR GROUP AND SYMMETRIC GROUP

1.1. Quick motivation and definitions. Let k be a field, and V a vector space over k . Then, the general linear group of V , denoted $GL(V)$, is the group of all linear automorphisms of V as a k -vector space. If V has finite dimension n over k , then we denote the group $GL(V)$ as $GL(n, k)$.

Given a set S , the symmetric group over S , denoted by $Sym(S)$, is defined as the group of all set-theoretic automorphisms of S . When S has cardinality n , this group is denoted by S_n or $Sym(n)$.

Here, we shall explore the question of the parallel between $GL(n, k)$ and S_n . In particular, we shall see that:

- The symmetric group S_n is the combinatorial *avatar* of $GL(n)$, obtained by *forgetting* the algebraic structure on it.
- Combinatorics/geometry for the general linear group corresponds to combinatorics/geometry for the symmetric group.

1.2. Flags in both. We first define flags in vector spaces, which correspond to the general linear group:

Definition. Let V be a vector space. A **flag**_(defined) in V is an ascending chain of subspaces of V . We assume that the chain begins with the empty subspace and ends at the whole space. A general flag looks like:

$$0 = V_0 \leq V_1 \leq V_2 \leq \dots \leq V_r = V$$

For an ordered integer partition $n = m_1 + m_2 + \dots + m_r$, a flag is said to be of type (m_1, m_2, \dots, m_r) if the dimension of V_i is $\sum_{j \leq i} m_j$ or equivalently, the dimension of V_i is the dimension of V_{i-1} plus m_i .

A **complete flag**_(defined) is a flag of type $(1, 1, 1, \dots, 1)$, or equivalently a flag:

$$0 = V_0 \leq V_1 \leq V_2 \leq \dots \leq V_n = V$$

where each V_i has dimension one more than V_{i-1} .

We can define an analogous notion for the symmetric group. The idea is to replace “subspaces” by “subsets” and the dimension of a space by the cardinality of a set.

Definition. A **flag for the symmetric group**_(defined) is an ascending sequence of subsets in it. We assume that the sequence begins with the empty set (the zeroth member) and ends with the whole space.

A flag is said to be of type m_1, m_2, \dots, m_r if the i^{th} member is m_i more than its predecessor (in size).

A complete flag in the symmetric group is an ascending sequence where the size of the subset increases by 1 at each step. thus, it is a flag of type $(1, 1, \dots, 1)$.

1.3. **Bases to flags.** Let k be a field and V be a n -dimensional vector space over k . As such, V does not have a natural choice of basis.

Given a particular ordered basis $\{v_1, v_2, \dots, v_n\}$ for V , we can construct an ascending chain of subspaces V_i where V_i is the linear span of $\{v_1, v_2, v_3, \dots, v_i\}$. We then have:

$$V_0 \leq V_1 \leq \dots \leq V_{n-1} \leq V_n$$

Here each V_i has dimension i , and is obtained from V_{i-1} by adding one more vector, viz v_i .

This prompts the definition:

Definition. The **complete flag associated with an ordered basis**_(defined) is the flag whose i^{th} member is the subspace generated by the first i elements of the ordered basis.

Correspondingly we can define, for any “ordered sequence” of distinct integers from 1 to n (which is simply a permutation in one-line notation) is an ascending sequence of subsets, obtained as follows:

Definition. The ascending sequence of subsets corresponding to an ordered sequence of distinct integers from 1 to n is the sequence where the i^{th} subset is the subset comprising the first i elements.

1.4. **Automorphism group and ordered bases.** The general linear group acts on the set of all ordered bases of the vector space, in a natural way. Now, the interesting thing is that this action of the general linear group is a regular group action. In other words, it is true that given any two choices of ordered basis, there is a unique element of the general linear group taking one to the other.

Thus, once we fix a certain choice of basis as our *standard* basis, then we can define a bijection from the general linear group to the set of standard bases, by sending each element of the general linear group to the image of the standard basis under the action.

So we have a bijection:

$$\text{General linear group} \leftrightarrow \text{Ordered bases}$$

A similar bijection exists for the symmetric group. Here, the symmetric group acts on the set of “one-line notation permutations”. Again, if we fix one permutation as the standard permutation (e.g. the permutation $123 \dots n$) then we can identify elements of the symmetric group with one-line notation permutations.

That gives us a bijection;

$$\text{Symmetric group} \leftrightarrow \text{Ordered sequences of distinct integers upto } n$$

1.5. **The relation between bases and flags.** From what we have seen above, there is a map:

$$\text{Ordered bases} \rightarrow \text{Complete flags}$$

which takes an ordered basis and gives the flag obtained by taking the span of initial segments of the ordered basis.

By the identification above, we can treat this as a map;

$$GL(n) \rightarrow \text{Complete flags in } n\text{-dimensional space}$$

The corresponding map for the symmetric group is:

$$S_n \rightarrow \text{Complete flags for the symmetric group}$$

Here, an important difference creeps in. In both cases, the map is surjective, viz every flag can be obtained from an element of the automorphism group. However, the map in the case of S_n is bijective. In other words, given a complete flag, we can recover a unique permutation which gave rise to it. This is because each time we enhance the subset’s size by 1, there is only one possibility for the “next” element in S_n .

However, the map in the case of $GL(n)$ is not injective. This is because when we enlarge the dimension of a subspace by 1, there are lots of possible vectors we could have added to obtain the bigger vector space. More precisely to go from V_i to V_{i+1} we could add any vector in $V_{i+1} \setminus V_i$.

We shall now see that the map from $GL(n)$ to the set of complete flags is a quotient map by the Borel subgroup B , which is the subgroup of upper triangular invertible matrices.

1.6. Explanation for two-by-two case. An ordered basis of a vector space gives rise to an ordered direct sum decomposition of the vector space into one-dimensional subspaces. For a basis v_1, v_2, \dots, v_n , we have:

$$V = kv_1 \oplus kv_2 \oplus \dots \oplus kv_n$$

On the other hand, the complete flag corresponding to this looks like:

$$0 \leq kv_1 \leq kv_1 \oplus kv_2 \leq \dots \leq kv_1 \oplus kv_2 \oplus \dots \oplus kv_n$$

For simplicity consider the case where $n = 2$. Then, we have that a basis for V looks like:

$$V = kv_1 \oplus kv_2$$

while the corresponding complete flag is:

$$0 \leq kv_1 \leq kv_1 \oplus kv_2 = V$$

We now try to determine conditions on another basis w_1, w_2 such that the complete flag corresponding to them is the same as the above.

Note the first step of this complete flag:

$$0 \leq kv_1$$

Instead of choosing v_1 itself, we could have chosen any other nonzero element in the one-dimensional subspace generated by v_1 .

Thus v_1 could be replaced by any element of the form av_1 where $a \neq 0$.

Note the second step of this complete flag viz the part:

$$kv_1 \leq kv_1 \oplus kv_2 = V$$

This involves *adding* the vector v_2 to enhance the one-dimensional subspace kv_1 to the two-dimensional space V .

Note that instead of v_2 , any other vector that does not lie inside kv_1 could have been added, to produce the same whole space. Thus, any element of the form $bv_1 + cv_2$ where $c \neq 0$, could do as the second basis vector.

Thus the upshot is that w_1, w_2 looks like:

- w_1 is of the form av_1 where $a \neq 0$
- w_2 is of the form $bv_1 + cv_2$ where $c \neq 0$

Writing the matrix taking v_1, v_2 to w_1, w_2 in standard basis, we obtain:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $ac \neq 0$

1.7. Explanation in higher-dimensional cases. The situation is a little more complex in the higher-dimensional case, but the idea is the same. Namely, if V_{i-1} is the span of vectors $v_1, v_2, v_3, \dots, v_{i-1}$, we want to add a new vector w_i that along with V_{i-1} generates V_i . It is clear that w_i must be of the form $\sum_{j \leq i} a_{ij}v_j$ where the coefficient of v_i is nonzero.

Thus, the matrix that transforms the basis v_1, v_2, \dots, v_n to w_1, w_2, \dots, w_n is an upper triangular matrix with all the diagonal entries nonzero.

The upshot: a linear transformation preserves the standard complete flag if and only if its matrix in the standard basis is upper triangular, with all the diagonal entries nonzero.

2. PARTIAL FLAGS

So far, we have seen that ordered direct sum decompositions into one-dimensional subspaces give rise to complete flags. More generally, we have the following.

Let $n = m_1 + m_2 + \dots + m_r$ and let $s_i = \sum_{j \leq i} m_j$. Then, a decomposition:

$$V = \bigoplus V_i$$

where V_i has dimension m_i gives rise to a partial flag:

$$0 \leq V_1 \leq V_1 \oplus V_2 \leq \dots \leq V_1 \oplus V_2 \oplus V_3 \oplus \dots \oplus V_n = V$$

Let $F_i = \bigoplus_{j \leq i} V_j$. Then the dimension of F_j is s_j .

Thus: direct sum decompositions corresponding to an ordered integer partition give rise to partial flags where the dimensions are the partial sums for the ordered integer partition.

2.1. Parabolic subgroups. Let $n = m_1 + m_2 + m_3 + \dots + m_r$ be an ordered integer partition. Then, the standard direct sum decomposition corresponding to this is the direct sum decomposition into subspaces where:

- The first direct summand is the subspace generated by the first m_1 basis vectors
- The next direct summand is the subspace generated by the next m_2 basis vectors
- And so on, till the last direct summand which is the subspace generated by the last m_r basis vectors

The standard partial flag is the partial flag corresponding to this. In other words, it is the one where the i^{th} vector space is the vector space generated by the first s_i basis vectors, where $s_i = m_1 + m_2 + \dots + m_i$.

Note that $GL(V)$ acts on the set of direct sum decompositions, and also on the set of partial flags, and moreover it preserves the type (viz, the corresponding ordered integer partition) for a flag. Thus, we can ask: what is the isotropy subgroup of the standard partial flag corresponding to the ordered integer partition $n = m_1 + m_2 + \dots + m_r$? it turns out that this is the subgroup described as follows:

Make square blocks of size m_1 followed by m_2 followed by m_3 and so on along the diagonal. Then, a matrix that stabilizes the standard partial flag must satisfy:

- The entries on the lower left side of these blocks must all be zero
- The part in each of the square block must form an invertible matrix
- The entries above can be arbitrary

The subgroup comprising all such matrices is termed the **standard parabolic subgroup**_(defined) for the ordered integer partition.

Extreme cases:

- The standard parabolic subgroup for the partition into all 1s is the Borel subgroup
- The standard parabolic subgroup for the partition into just n itself, is the whole group

2.2. Partial flags and parabolics for the symmetric group. In the symmetric group, a partial flag is just a flag viz an ascending sequence of subsets, where the sizes could grow by more than one.

2.3. Summary so far. For the general linear group, we have the following:

- Upto a choice of standard basis, the general linear group can be identified with the set of ordered bases.
- There is a natural map from the set of ordered bases to the set of complete flags. This gives a map from the general linear group to the set of complete flags.
- The map from the general linear group to the set of complete flags is the quotient map by a subgroup, called the Borel subgroup, which, when written in matrix form, is the subgroup of upper triangular matrices.
- Given any ordered integer partition of n , there is a map from ordered direct sum decompositions of that type to flags of that type.
- The isotropy subgroup for the standard flag of a given type is the parabolic subgroup of that type, which is obtained as matrices given by a “staircase” description. The space of flags of that type can be viewed as the quotient of the whole space by this parabolic subgroup.

For the symmetric group, we have the following:

- Upto a choice of “standard permutation”, the symmetric group can be identified with the set of ordered sequences of distinct members

- There is a natural map from the set of ordered sequences (ordered *bases*) to complete flags (ascending sequences of subsets growing by 1 at a time).
- This gives a map from the symmetric group to the set of ascending sequences of subsets. The map turns out to be a bijection.
- Given any ordered integer partition of n , there is a map from ordered partitions into subsets whose sizes correspond to that partition, to ascending sequences of subset whose type is that partition. The map turns out to be a bijection.
- The parabolic subgroup for a partition is the same as the Young subgroup, viz the parabolic for a partition $n = m_1 + m_2 + \dots + m_r$ is the same as $S_{m_1} \times S_{m_2} \times \dots \times S_{m_r}$.

3. DOUBLE COSETS AND RELATIVE POSITIONS OF FLAGS

3.1. Matrix of intersection numbers. We now explore the question: given two flags F and F' , what are the possible *relative positions* they could be in? We first introduce an important notion, the matrix of intersection numbers, for two flags.

We assume that all flags begin at 0 and end at the whole space.

Let F be a flag of length r and F' be a matrix of length s . Denote by F_i the i^{th} member of F and by F'_j the j^{th} member of F' . Then the matrix of intersection numbers of F and F' is the matrix whose ij^{th} entry is:

$$\frac{\dim(F_i \cap F'_j)}{\dim(F_i \cap F'_{j-1} + F_{i-1} \cap F'_j)}$$

A little playing around will show us that:

- The sum of the entries in row i is the dimension of F_i minus the dimension of F_{i-1}
- The sum of the entries of column j is the dimension of F'_j minus the dimension of F'_{j-1}
- All the entries are nonnegative integers
- The sum of all the entries is n

In particular, for complete flags, the matrix that we get is a permutation matrix.

3.2. Why this describes the relative position.

Definition. Consider the action of $GL(V)$ (diagonally) on ordered pairs of flags (viz the space $G/B \times G/B$). Then an orbit under this action is termed a **relative position of flags**_(defined). Thus (F, F') is in the same relative position as (F', F'') if they are in the same orbit under the action of G .

From general group theory, we know that the set of orbits described above is the same as the double coset space of G by B . We replace each orbit by its intersection with those pairs of flags where the first member is B (corresponding to the standard flag). Then, for each orbit, the choices for the second member correspond to one double coset.

We now claim that the matrix of intersection numbers provides is a complete invariant describing the relative position of two matrices. Equivalently, consider the map:

$$GL(V) \rightarrow S_n$$

that sends each matrix g to the relative position of the standard flag and g times the standard flag.

We claim that the fibres of this map are precisely the double cosets. In other words, the elements of the symmetric group form a system of double coset representatives for the Borel subgroup, or equivalently, we have:

$$GL(n) = \bigcup_{w \in S_n} BwB$$

Here is a rough outline of the proof: we use the matrix of intersection numbers to construct an ordered basis for both the flags, such that the transformation matrix from one basis to the other is simply the permutation matrix.

3.3. A more general construct of maps. We have so far seen that there exists a map:

$$GL(n) \rightarrow S_n$$

which sends each matrix $g \in GL(n)$ to the relative position of the standard flag and g times the standard flag.

Further, we saw that the fibres of this map are double cosets of the Borel subgroup.

Note that the map is not a group homomorphism, and nor are the fibres of equal size. However, this map does exhibit some regularity features:

- It is preserved by the concatenation operation. In other words, the diagram below commutes:

$$\begin{array}{ccc} GL(m) \times GL(n) & \rightarrow & GL(m+n) \\ \downarrow & & \downarrow \\ S_m \times S_n & \rightarrow & S_{m+n} \end{array}$$

- The inverse image of the identity element is a subgroup

In fact, it is in general true that a lot of the structure that we have for the symmetric group can, in principle, be extended to the general linear groups, with maps coming from the structures in the general linear group to their counterparts in the symmetric group.

The general philosophy is as follows:

For each combinatorial structure associated with the symmetric group, define an analogous combinatorial structure for the general linear group, and a map from the structure corresponding to the general linear group to the structure corresponding to the symmetric group. Further, for every map between combinatorial structures at the level of the symmetric group, define a corresponding map at the level of the general linear group that “lifts” this map.

3.4. The matrix of intersection numbers. We earlier saw a description of the matrix of intersection numbers of two arbitrary flags. We now use this to define a map from the collection of flags of a given type, to permutations of that type.

- Construct the matrix of intersection numbers of the standard complete flag and the given flag. If the flag is of type (m_1, m_2, \dots, m_r) , this is a $n \times r$ matrix with each column containing exactly one 1 and each row containing m_i ones.
- We now construct a flag in the symmetric group as follows: let A_1 be the set of positions from 1 to n that have a 1 in the first row. let A_2 be the set of positions from 1 to n that have a 1 in the first two rows, and so on. Then the flag is: $\emptyset \leq A_1 \leq A_2 \leq \dots \leq A_n = \{1, 2, \dots, n\}$

3.5. The individual fibres. We have this picture:

- There is a set of all flags in the general linear group
- This set can be partitioned based on types
- There is a map from the set of all flags of a particular type in the general linear group to the set of flags of the same type in the symmetric group.

We are now interested in the fibres of this map. In other words, we are interested in the inverse images of points for the following map:

$$\text{Flags of type } (m_1, m_2, \dots, m_r) \text{ in } GL(n, k) \rightarrow \text{Flags of type } (m_1, m_2, \dots, m_r) \text{ in } S_n$$

Equivalently, given an ascending sequence of subsets of $\{1, 2, \dots, n\}$ with sizes $m_1, m_1 + m_2$ and so on, we want to determine the set of all flags in the general linear group that map to this particular ascending sequence of subsets.

4. TWO EXTREME CASES

4.1. The two cases. Among all flag types, we are interested in two extreme cases:

- The complete flags. The set of all complete flags, which is G/B , can be partitioned as a union of cells based on the inverse images of permutations (complete flags for the symmetric group).

- The flags of length 2 viz things of the form:

$$0 \leq W \leq V$$

There are $n - 1$ possible types to these, depending on the dimensions of W . For the dimension of W equal to m , the corresponding collection of flags is called the Grassmannian $Gr(n, m)$ and is equal to the quotient of G by a suitable parabolic subgroup.

Further each Grassmannian variety can be partitioned as a disjoint union of Schubert cells, where a Schubert cell is the inverse image of a flag of length 2 for the symmetric group.

4.2. A closer analysis of the Grassmannians. For the Grassmannian of subspaces of dimension m , the general map we discussed above becomes:

$$Gr(n, m) \rightarrow \binom{\{1, 2, \dots, n\}}{m}$$

In other words, we have a map from m -dimensional subspaces to subsets of size m .

Now a natural question is: given a particular choice of a subset of size m , what are the m -dimensional subspaces that give rise to that subset?

Let's look at one-dimensional subspaces of a three-dimensional space. The matrix corresponding to the one-dimensional space could have as its first row 001 or 010 or 100.

(the second row will have 0s below the 1s and 1s below the 0s).

Now, if you imagine picking an arbitrary line in k^3 , then:

- There's only one way it can give 100: if it exactly coincides with the vector space spanned by the first basis vector. This is, in general, highly unlikely
- It gives 010 only if it lies in the vector space spanned by the first two basis vectors (but does not coincide with the vector space spanned by the first basis vector). This, too is highly unlikely, specially for a field with lots of elements.
- In all other cases, it gives 001.

In general, picking an arbitrary subspace, or a subspace in fairly general position, is likely to give us something that does not intersect any of the first $n - m$ members of the flag nontrivially. Thus, the first row of its matrix looks like a bunch of zeroes followed by a bunch of ones, while the second row looks like a bunch of ones followed by a bunch of zeroes.

4.3. More on Schubert cells and Schubert varieties. Recall that a Schubert cell is the set of subspaces of dimension m that map to a particular subset of size m . It turns out that each Schubert cell is isomorphic to a vector space over the base field. The reason for this is roughly similar to the argument we saw for the Borel subgroup being the subgroup of upper triangular matrices. Namely, given a subset of size m , we can parametrize the subspaces that map to it by means of "upper triangular" coordinates.

Let's look at the example of 1-dimensional subspaces of 3-dimensional space. Consider the 1-dimensional subspaces that map to 001. For these subspaces, any basis vector that we choose must be a linear combination of the three standard basis vectors, and the coefficient of the third standard basis vector must be nonzero. Dividing out by this coefficient, we get the unique basis vector where the third coefficient is 1. The first two coefficients both vary freely over k . Thus this Schubert cell is the same as k^2 .

In fact, if we are looking at the Grassmannian of 1-dimensional subspaces in n -dimensional space, the inverse image of the 1-dimensional subset i is the vector space k^{i-1} .

For finite fields of size q , this also gives a proof of the identity:

$$\frac{q^n - 1}{q - 1} = 1 + q + q^2 + \dots + q^{n-1}$$

where the LHS is the computation of the Grassmannian using the binomial coefficient-type argument, and the RHS computes the cardinality as the sum of cardinalities of Schubert cells.

For other Grassmannians we get other interesting combinatorial identities.

4.4. Relation between Schubert cells. The Schubert cell of one-dimensional subspaces corresponding to 001 is dense in the Grassmannian of 1-dimensional spaces, and its closure contains all the other Schubert cells. Similarly, the Schubert cell corresponding to 010 has the Schubert cell for 100 in its closure, though not the third one. The Schubert cell corresponding to 100 is a closed point.

In general, we get a partial ordering on the Schubert cells (corresponding to any n , and any ordered integer partition of n). For instance, we also get a partial ordering on the Schubert cells corresponding

to complete flags. Since the Schubert cells correspond to elements of the symmetric group, we obtain a partial ordering on the symmetric group. This partial ordering is called the Bruhat ordering and it has a lot of combinatorial significance.

5. RELATED WORK AND STUDY

5.1. Convolution product and Hecke algebras. The construction described in this section works when the underlying field is finite.

We have a map:

$$GL(n) \rightarrow S_n$$

as we noticed earlier, this map is *not* a group homomorphism. In fact, it is a very strange map, in the sense that *most* elements map to the longest element of S_n (this is the permutation which in one-line notation is $n(n-1)\dots 1$).

However, we can define a certain multiplication and create an algebra structure on these that makes them into a variant of the symmetric group. This structure is the so-called Hecke algebra structure. Here is the description:

- Let R be any ring. Consider the set of all functions from S_n to R . Denote this set as H_n . A natural basis for H_n is the characteristic functions for each element of S_n .
- Define a multiplication operation on H_n on the basis elements (and then extend linearly) as follows. Let $\sigma, \sigma', \sigma''$ be three permutations. Then the coefficient of the characteristic function for σ'' in the product of those for σ and σ' is obtained as follows. Let (P, Q) be a pair of flags in relative position σ'' . Then the coefficient is the number of flags R such that the relative position of (P, R) is σ and the relative position of (R, Q) is σ' .

This convolution product gives an associative algebra structure, and it is this associate algebra that we dub the Hecke algebra. It turns out that all the coefficients for the multiplication can be expressed as polynomials in q , where q is the order of the field. This allows us to try to put values of q which do not occur as orders of fields, and get “Hecke algebras” that do not actually arise from any field.

Further, when we put $q = 1$, we recover the group algebra of the symmetric group.

5.2. Weyl groups in greater generality. Recall the map:

$$GL(n) \rightarrow S_n$$

In some sense, what this map does is to *remove* the algebraic structure from $GL(n)$ and leave *only* the combinatorial structure. This intuitive idea is justified by the explanation that the image of the map is independent of the underlying field, and carries only the “combinatorial information” about the flag.

We can define analogous maps for other algebraic and Lie groups. The general idea is:

- We define a notion of flag for the given algebraic or Lie group. The flag is an ascending chain of vector spaces that may preserve some further structure. We also define a standard flag
- We define the Borel subgroup B of the algebraic/Lie group as the subgroup that stabilizes the standard flag
- We then define a relative position map of two flags, and show that this relative position map parametrizes the double cosets. Further, the images of the relative position map form a group, called the Weyl group.
- For finite fields, we can also define a Hecke algebra using a convolution multiplication similar to the one for the general linear group. Again, the coefficients for all the multiplication are polynomials in the order of the field, and when we put that value as 1, we recover the group algebra of the Weyl group.

Let’s look at the orthogonal group in even dimensions over complex numbers. This is the stabilizer of the bilinear form:

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Note that over the reals, this is actually equivalent to the split-orthogonal group, and not to the ordinary orthogonal group in general. However, all orthogonal groups are equivalent over complex numbers, because there is only one equivalence class of symmetric bilinear forms.

- The complete flags here are ascending sequences of subspaces, with some special orthogonality properties between the subspace at stage i and the subspace at stage $n + i$.
- The Weyl group is the group of signed permutation matrices of order n , with overall determinant 1. We can blow this into a $2n \times 2n$ matrix by replacing each 1 by the identity 2×2 matrix and a -1 by the matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In fact, the matrix in this form is what we get if we try to compute it as a matrix of intersection numbers.

5.3. Lie algebras and root systems. To any Lie group (differential manifold with a group structure) we can associate a corresponding Lie algebra. The Lie algebra can be viewed in the following equivalent ways:

- As the left-invariant vector fields on the Lie group
- As the right-invariant vector fields on the Lie group
- As the tangent space at the origin

We define a bracket on the Lie algebra, called the Lie bracket, which can be thought of as “differentiating” the commutator operation in the Lie group.

To each smooth representation of the Lie group, we obtain a Lie algebra-theoretic representation of its Lie algebra.

We now describe the construction/meaning of root spaces, roots, and root systems:

- To each Lie algebra, we can associate a Cartan subalgebra, which is a maximal subalgebra subject to the property that the Lie bracket of any two vectors in it is zero
- Given a representation of a Lie algebra, we get a representation of the corresponding Cartan subalgebra. We try to decompose the representation space as a direct sum of what are called weight spaces. A weight space is a subspace that is simultaneously an eigenspace for the action of all vectors in the Cartan subalgebra. The map from the Cartan subalgebra to the eigenvalues is an element in the dual space to the Cartan subalgebra, and this is called a weight vector.
- In the particular case when the representation is the adjoint representation of the Lie algebra on itself, the weight spaces are called weight vectors, and the weight vectors are termed roots. The set of all roots is said to form a root system, and it satisfies some properties of an abstract root system.
- In this root system, we can talk of positive roots, and simple roots, We can pick a collection of simple roots, and encode the information about angles between them in a diagram called a Coxeter diagram or a Dynkin diagram.
- The Weyl group can then be defined as the Coxeter group corresponding to this Coxeter diagram.

5.4. Euler characteristics. We had earlier seen that it is not directly possible to interpret the map from $GL(n)$ to S_n as a group homomorphism, nor is it directly possible to give a structure on the double cosets that is the same as the group structure on S_n . In the finite case, we used a convolution product and showed that it is a deformation of the product in the symmetric group. There is another approach that works in the topological case, say for complex numbers.

Here, we define the product of permutations σ and σ' as the unique permutation σ'' such that given flags F and F'' in relative position σ'' , the space of flags F' such that the relative position of (F, F') is σ and that of (F', F'') is σ' , has Euler characteristic 1. For all other choices of σ'' , the Euler characteristic turns out to be zero.

This can be thought of as a convolution product where it turns out that a convolution of point measures is a point measure.

5.5. Quantum groups. Start with a nice algebra, such as the group algebra, and then try to “deform” it using a deformation parameter, such that setting the deformation parameter equal to 1 recovers the original group algebra. We saw this idea being used for the Hecke algebra. The general one-parameter Hecke algebra of order n , with parameter q , is such that:

- When we set $q = 1$, we recover the group algebra of the symmetric group of order n
- When we set q as the order of a finite field, we recover the Hecke algebra of order n over the finite field

As the Hecke algebra can be viewed as deformations of the group algebra of S_n , we can, in the same way, consider the deformations of group algebras of general linear groups. These deformations are called quantum groups. There are two approaches to this: the construction by Drinfeld and the construction by Jimbo.

5.6. Some nice correspondences in representation theory. Here are a few nice facts in representation theory:

- (1) The representation theory of the symmetric group over a finite field F_q is closely related to the representation theory of the Hecke algebra, specialized to a q^{th} root of unity.
- (2) The representation theory of $GL_n(F_q)$ over the complex numbers is closely related to the representation theory of the quantum group for $GL_n(\mathbb{C})$, specialized to a q^{th} root of unity.

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