ARTIN'S THEOREM: FOR THE STUDENT SEMINAR

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ABSTRACT. This is a write-up of Artin's theorem for the student seminar in the course on Representation Theory of Finite Groups. Much of the material is taken from Serre's book on representation theory.

1. Notation and motivation

1.1. Ring of functions on a finite group.

First, some basic observations:

- Given a set and a ring, the set of functions from the set to the ring forms a ring forms a ring under pointwise addition and pointwise multiplication.
- Given a set and an equivalence relation on it, the set of functions constant on the equivalence classes forms a subring of this ring.

We are interested in a setup where the set in question is a finite group, the equivalence relation isthat of beig conjugate, and functions are being taken to the complex number $\mathbb{C}$. By the natural ring structure on $\mathbb{C}$, the set of all functions from G to $\mathbb{C}$ has a ring structure. Further, the ring of functions constant on the cnojguacy classes (that we shall call class functions) is a subring of this. We shall denote this subring as $F_C(G)$.

$F_C(G)$ is a ring, in fact it is a $\mathbb{C}$-algebra. It is also a $\mathbb{C}$-vector space whose dimension is the number of conjugacy classes of G.

1.2. Characters.

Let $\rho : G \to GL(V)$ be a representation where V is a complex vector space. Let χ be the character of ρ.

Then, $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$ Thus, $\rho(g)$ and $\rho(hgh^{-1})$ are conjugaet in $GL(V)$. Hence, they have the same trace. Thus, $\chi(g) = \chi(hgh^{-1})$. Hence χ is a class function.

Let's now recall some basic facts about representation theory of finite groups over $\mathbb{C}$.

- Every representation can be expressed uniquely as a direct sum of irreducible representations. Thus, the character of any representation can be expressed as a $\mathbb{N}$-linear combinations of the irreducible characters.
- The irreducible characters are linearly independent and form a $\mathbb{C}$-basis for the $\mathbb{C}$-vector space $F_C(G)$.
- The character uniquely determines the representation.
- The product of two characters is a character. In fact, it is the character of the tensor product of the corresponding representations.

We denote the collection of all characters as $R^+(G)$. From the above facts, $R^+(G)$ is closed under addition and multiplication. However, by the linear independence, $R^+(G)$ is not closed under negation.

We now make another definition:

Definition (Virtual characters)**.** A **virtual character**(defined) over $\mathbb{C}$ is a function that can be expressed as the pointwise difference of two characters. The virtual characters form a ring, which we denote as $R(G)$.

1.3. Generating sets for $R(G)$.

$R(G)$ is a ring and hence also a $\mathbb{Z}$-module. In fact, it is a finitely generated free $\mathbb{Z}$-module, with basis being the irreducible characters. We want to explore two questions:

- Can we get a collection of characters induced from subgroups such that $R(G)$ is generated as a $\mathbb{Z}$-module by them?

- Can we get a collection of character induced from subgroups such that $R(G)$ is in the $\mathbb{Q}$-module generated by them?

We are interseted in these questions because we would like to understand the theory of character theory of G from the character theory of "small" subgroups of G.

The best results for the two questions:

- **Brauer's theorem**: This states that the collection of characters induced from elementary subgroups generates $R(G)$ as a $\mathbb{Z}$-module.
- **Artin's theorem**: This states that the $\mathbb{Q}$-module generated by characters induced from cyclic subgroups contains $R(G)$.

1.4. **Full statement of Artin's theorems.**

Theorem 1 (Artin's theorem)**.** Let X be a collection of subgroups of a finite group G. Then the following are equivalent:

- The union of the conjugates of subgroups contained in X is the whole of G.
- Every character of G is a rational linear combination of characters induced from subgroups contained inside X.

Noet that if we consider the collection of cyclic subgroups, then clearly since every element of G lies inside a cyclic subgroup, the union of all cyclic subgroups of G is the whole of G. Thus, it follows that every character of G is a $\mathbb{Q}$-linear combination of characters induced from cyclic subgroups.

1.5. **Some remarks on $R(G)$ and $F_C(G)$.** We'll begin with an easy lemma that follows from matrix theory:

Lemma 1. If a subset of $R(G)$ generates $F_C(G)$ as a $\mathbb{C}$-vector space, then $R(G)$ is contained in the $\mathbb{Q}$-vector space spanned by it.

Thus, Artin's theorem is equivalent to the following:

Theorem 2 (Artin's theorem)**.** Let X be a collection of subgroups of a finite group G. Then the following are equivalent:

- The union of the conjugates of subgroups contained in X is the whole of G.
- The class functions induced from class functions on subgroups contained inside X, generate the space of class functions on G.

2. Proof of Artin's theorem

2.1. **Quick recall of the definition of induction.** I will give one of many equivalent definitions of induction for a class function. Let $H \leq G$ be groups. Let S be a system of coset for H in G. Let ϕ be a class function on H. The, we define the $\mathsf{Ind}_H^G(\phi)$ as the following function

$$x \mapsto \sum_{s \in S} \phi_0(sxs^{-1})$$

where ϕ_0 is a function that coincides with ϕ inside H and is 0 outside.

2.2. **On what elements is the induction nonzero?** Here are some easy observations:

(1) $\mathsf{Ind}_H^G(\phi)$ can be nonzero at x only if there is at least one s such that sxs^{-1} lies inside H. Thus, $\mathsf{Ind}_H^G(\phi)$ is zero *off* the conjugates of H.

When ϕ is taken to be the trivial character, the converse also holds: $\mathsf{Ind}_H^G(\phi)$ is zero precisely on those elements that are not contained in the conjugates of H.

(2) $\mathsf{Ind}_H^G(\phi)$ is zero outside H if H is normal in G.
Again, if ϕ is the trivial character, the converse holds: H is normal if and only if $\mathsf{Ind}_H^G(\phi)$ is zero outside H.
(3) If H is normal and has the property that every inner automorphism of G takes elements of H to their H-conjugates, then $\mathsf{Ind}_H^G(\phi)$ is $[\ G\ :\ H\ ]$ times ϕ on H and 0 outside.

Thus, it is clear that characters induced from H can only generate class functions on the union of conjugates of H.

2.3. **Direct sum of images under induction.** Recall the setup for Artin's theorem: X is a family of subgroups of G, and we are interested in determining the $\mathbb{Q}$-linear combinations of characters induced from subgroups inside H. First, we observe that for every H, we have a map:

$$\mathsf{Ind} : F_C(H) \to F_C(G)$$

This map is a homomorphism of $\mathbb{Z}$-modules because of the following facts:
We are interested in sum of the images of $F_C(H)$ for all $H \in X$.

2.4. **Induction and restriction: adjoint functors.** Let's haev a quick look at the Frobenius reciprocity:

Theorem 3 (Frobenius reciprocity)**.** Let $H \leq G$ be groups, and ϕ, ψ be class functions on H and G respectively. Then:

$$\left\langle\ \mathsf{Ind}_H^G(\phi)\ ,\ \psi\ \right\rangle = \left\langle\ \phi\ ,\ \mathsf{Res}_H^G(\psi)\ \right\rangle$$

The Frobenius reciprocity is often viewed as saying induction and restriction are *adjoint* functors.
An important consequence of Frobenius reciprocity is the following:

Claim. Let $H \leq G$ be groups. A class function on G restricts to the zero function on H if and only if it is orthogonal to the space generated by all class functions induced from H.

Another variant of the claim:

Claim. Let X be a collection of subgroups of G. A class function on G restricts to the zero function on every member of X if and only if it is orthogonal to every class function induced from H.

2.5. **The two maps.** Recall that we wanted to prove that if X is a collection of subgroups of G, the following are equivalent:

- The union of the conjugates of subgroups contained in X, is the whole of G.
- The $\mathbb{C}$-vector space generated by the images of inductions from $F_C(H)$ for $H \in X$, is the whole of $F_C(G)$.

The second statement can be translated to the following: the sumof the images of maps $\mathsf{Ind} : F_C(H) \to F_C(G)$ for $H \in X$, is the whole of G. In other words, the natural map:

$$\mathsf{Ind} : \bigoplus_{H \in X} F_C(H) \to F_C(G)$$

defined by induction is surjective.
We shall consider another map:

$$\mathsf{Res} : F_C(G) \to \bigoplus_{H \in X} F_C(H)$$

that sends $\chi \in F_C(G)$ to the tuple whose $F_C(H)$ coordinate is $\mathsf{Res}_H^G(\chi)$.

We now make the following claim:

Lemma 2. Res is injectiv e if and only if the image of Ind generates the whole of $F_C(G)$ as a $\mathbb{C}$-vector space.

Proof. From the claim made earlier, a class function on G restricts to 0 on all the member of X if and only if it orthogonal to the image of the map Ind. Thus, if Res is injective, the orthogonal space to the image of Ind is trivial. This means that the subspace generated by the image of Ind is the whole space. □

And now the final claim:

Claim. If the union of the conjugates of member of X is the whole of G, then Res is injective.

Proof. Let $f \in F_C(G)$ such that the restriction of f to any member of X is zero. We want to show that given any $g \in G$, $f(g) = 0$.

There exists a conjugate g' of g and a subgroup $H \in X$ such that $g' \in H$. Since f is a class function, $f(g') = f(g)$. But $f(g') = 0$, and hence $f(g) = 0$. □

2.6. **Piecing together the proof.** Let X be a collection of subgroups of G. We have shown that all the following are equivalent:

(1) The union of conjugates of members of X is the whole of G.
(2) Any class function on G that restricts to the zero function on all members of X, is itself the zero function.
(3) Every class function on G is a $\mathbb{C}$-linear combination of characters induced from member of H.
(4) Every character of G is a $\mathbb{Q}$-linear combination of characters induced from members of H.

3. Examples and counterexamples

3.1. **$\mathbb{Z}$-linear combinations: motivation for Brauer's theorem.** A natural question is: can all characters of G be represented as $\mathbb{Z}$-linear combinations of characters induced from cyclic subgroups?

The answer is clearly negative even in the case of Abelian groups. For this purpose, we recall an earlier remark. We begin with a definition:

Definition. A subgroup of a group is termed **conjugacy-closed**(defined) if whenever two elements of the subgroup are conjugate in the group, they are conjugate in the subgroup as well.

we had earlier remarked that if f is a class function on a conjugacy-closed normal subgroup of index r, then f induced to the whole group is 0 outside the subgroup and r times f inside the subgroup.

Note that any direct factor is a conjugacy-closed normal subgroup. Now, consider the cyclic group $\mathbb{Z}/2\mathbb{Z}$. It has two characters: the trivial character and the sign character. Consider the group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and X is the collection of cyclic subgroups of G. Then, any character induced from a subgroup in X must take *even* values at all elements of G. Hence, any $\mathbb{Z}$-linear combination of such characters must also be evenly valued. Thus, all characters of G are not covered in the $\mathbb{Z}$-linear combinations of characters induced from cyclic subgroups.

Let us try to understand the difficulty more closely.

In the above situation, what is happening is that characters induced from cyclic subgroups of G all take their values inside a proper ideal of the ring of all character values. This can, in turn, be traced to the problem that the indices of all cyclic subgroups have a nontrivial gcd. To rememdy this problem, we need to take a collection of subgroups wherein it is guaranteed that the gcd of the indices of the subgroups is trivial. Brauer's theorem, which looks at p-elementary subgroups, is a successful effort in this direction.

3.2. **Other instances of Artin's theorem.** Note that insetad of taking X as the collection of all cyclic subgroups, we could take other possibles for X. In fact, we could take only one representative for each conjugacy class of cyclic subgroups.

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