# PERMUTATION-EXTENSIBLE: A ROUTE TO EXTENSIBLE AUTOMORPHISMS

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Abstract. This article describes the use of permutation representations to obtain a partial solution to the Extensible Automorphisms Problem. It introduces a notion of permutation-extensibility and proves the equivalence of this notion with the notion of being subgroup-conjugating.

## 1. The problem we want to solve

### 1.1. Three big problems.

A quick recall of three definitions:

**Definition.** (1) An automorphism $\sigma$ of a group $G$ is termed **extensible**(defined) if for any embedding $G \leq H$, there is an automorphism $\sigma'$ of $H$ such that $\sigma'|_G = \sigma$.

For a particular embedding $G \leq H$, the group of automorphisms of $G$ that can be extended to automorphisms of $H$ is denoted as $\texttt{Extensible}(\ G\ ;\ H)$. The group of extensible automorphisms of $G$ is thus given by:

$$\texttt{Extensible}(G) = \bigcap_{G \leq H} \texttt{Extensible}(\ G\ ;\ H)$$

(2) An automorphism $\sigma$ of a group $G$ is termed **pushforwardable**(defined) if for any homomorphism $\rho : G \to H$ there is an automorphism $\sigma'$ of $H$ such that $\rho \circ \sigma = \sigma' \circ \rho$.

For a particular homomorphism $\rho : G \to H$, the group of automorphisms of $G$ that can be pushed forwadr via $\rho$ is denoted as $\texttt{Pushforwardable}(\ G\ ;\ \rho)$. The group of pushforwardable automorphisms is thus given by:

$$\texttt{Pushforwardable}(G) = \bigcap_{\rho : G \to} \texttt{Pushforwardable}(\ G\ ;\ \rho)$$

(3) An automorphism $\sigma$ of a group $G$ is termed **quotient-pullbackable**(defined) if for any surjective homomorphism $\rho : H \to G$ there is an automorphism $\sigma'$ of $H$ such that $\rho \circ \sigma' = \sigma \circ \rho$.

For a surjective homomorphism $\rho : H \to G$, the group of automorphisms of $G$ that can be *pulled back* to automorphisms of $H$ is denoted as $\texttt{Pullbackable}(\ G\ ;\ \rho)$. The group of quotient-pullbackable automorphisms of $G$ is thus gievn by:

$$\texttt{Quot} - \texttt{Pullbackable}(G) = \bigcap_{\rho : H \to G} \texttt{Pullbackable}(\ G\ ;\ H)$$

Some easy facts:

**Observation 1.**

- Any inner automorphism of a group is both pushforwardable and quotient-pullbackable. In symbols:

$$\mathsf{Inn}\,(G) \leq \mathtt{Quot-Pullbackable}\,(G)$$

and

$$\mathsf{Inn}\,(G) \leq \mathtt{Pushforwardable}\,(G)$$

- Any pushforwardable automorphism is extensible.

$$\mathtt{Pushforwardable}\,(G) \leq \mathtt{Extensible}\,(G)$$

It is *conjectured* that for all groups, the properties of an automorphism being inner, extensible, pushforwardable, and quotient-pullbackable, are all equivalent.

Given a group, we can *measure the extent* to which this conjecture holds by asking the following questions:

**Points of Investigation 1.**

- What is the group of extensible automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\mathsf{Inn}\,(G) = \mathtt{Extensible}\,(G)$?
- What is the group of pushforwardable automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\mathsf{Inn}\,(G) = \mathtt{Pushforwardable}\,(G)$?
- What is the group of quotient-pullbackable automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, in $\mathsf{Inn}\,(G) = \mathtt{Pushforwardable}\,(G)$?

1.2. **Where permutation representations take us.** Permutation representations give partial answers to only two of these three questions, namely, they impose conditions on extensible and pushforwardable automorphisms. Permutation representations do not help us with the problem of quotient-pullbackable automorphisms. Intuitively, this follows from the fact that the generic permutation group has only one proper nontrivial normal subgroup, and the quotient by that is always $\mathbb{Z}/2\mathbb{Z}$.

A little definition first:

**Definition** (Subgroup-conjugating automorphism)**.** An automorphism $\sigma$ of a group $G$ is termed **subgroup-conjugating**(defined) if, for any subgroup $H \leq G$, $\sigma(H)$ is a conjugate subgroup to $H$.

The main result we show is:

**Theorem 1** (Extensible automorphisms are subgroup-conjugating)**.** Let $G$ be a finite group and $\sigma$ an extensible automorphism of $G$. Then, given any subgroup $H$ of $G$, $\sigma(H)$ must be a conjugate subgroup to $H$ (In other words, $\sigma$ is a subgroup-conjugating automorphism).

## 2. Permutation representations

### 2.1. **Permutation-extensibility and permutation-pushforwardability.**

**Definition** (Permutation-extensible automorphism)**.** An automorphism $\sigma$ of a group $G$ is termed **permutation-extensible**(defined) if given any faithful permutation representation $\rho : G \to Sym(S)$ for a set $S$, $\sigma$ can be lifted to an *inner* automorphism of $Sym(S)$. The element of $Sym(S)$ corresponding to this inner automorphism is in fact precisely the *relabelling permutation* which changes the action of $G$ to the action of $G$ twisted by $\sigma$.

An automorphism $\sigma$ of a group $G$ is termed **permutation-pushforwardable**(defined) if given any permutation representation $\rho : G \to Sym(S)$ for a set $S$, $\sigma$ can be pushed forward via $\rho$ to an inner automorphism of $Sym(S)$. The element of $Sym(S)$ corresponding to this inner automorphism is in fact precisely the *relabelling permutation* which changes the action of $G$ to the action of $G$ twisted by $\sigma$.

Let's make some quick and easy claims:

**Lemma 1. ??** The following are equivalent for an automorphism $\sigma$ of a finite group $G$:

(1) $\sigma$ is permutation-extensible
(2) $\sigma$ is permutation-pushforwardable
(3) $\sigma$ can be pushed forward to an inner automorphism for all transitive permutation representations.

*Proof.* (2) $\implies$ (1), (2) $\implies$ (3): obvious

(3) $\implies$ (2): Suppose $\sigma$ can be pused to an inner automorphism for every transitive permutation representation. Consider a permutation representation $\rho$ on a set $S$. $\rho$, restricted to each orbit in $S$, is a transitive permutation representation, and hence, $\sigma$ can be pushed forward to an inner automorphism for the restriction to each orbit. Composing all these inner automorphisms gives the pushforward of $\sigma$ via $\rho$.

(1) $\implies$ (2): Suppose $\sigma$ is a permutation-extensible inner automorphism on a group $G$. Let $\rho$ be a permutation representation of $G$ on a set $S$. Suppose $\rho$ is not faithful. Then clearly $\rho$ cannot contain the regular representation as a component. Let $\tilde{\rho}$ be the direct sum of $\rho$ and the regular representation. This is a faithful representation, hence $\sigma$ must extend to an inner automorphism via $\tilde{\rho}$. Call the lift $\sigma'$.

$\sigma'$ is now a permutation on $S \sqcup G$ (here $G$ is the *set* acted upon as per the regular representation). Note that $\sigma'$ must preserve the size of orbits. Hence, no orbit inside $S$ can go to the orbit $G$. Thus, $\sigma'$ restricts to a permutation on $S$, and so $\sigma$ pushes forward to a permutation on $S$. □

Another easy claim, using the fact that for all $n \neq 6$, every automorphism group of the symmetric group $S_n$ is inner:

**Lemma 2.** Any extensible automorphism of a finite group is permutation-extensible.

*Proof.* Let $G$ be a finite group. Since each orbit of a permutation representation of $G$ is finite, it suffices to consider the case of finite faithful permutation representations of $G$. Suppose $\rho : G \to S_n$ is a finite faithful permutation representation of $G$. Let $\sigma$ be an extensible automorphism of $G$.

If $n \neq 6$: $\sigma$ extends to an automorphism of $S_n$, and since all automorphisms of $S_n$ are inner, $\sigma$ lifts to an inner automorphism.

If $n = 6$: Take a direct sum with any other representation, and argue. □

2.2. **Cosets and their stabilizers.** I recall facts from group actions that are useful here:

**Definition** (Self-normalizing subgroup)**.** A subgroup of a group is termed **self-normalizing**(defined) in the group if it equals its normalizer in the whole group.

**Fact 1** (Cosets and stabilizers)**.** Let $H \leq G$ be a subgroup. Then, for $g \in G$, the stabilizer of the coset $gH$ is the subgroup $gHg^{-1}$. Thus, there is a map from the space of left cosets of $H$ to the set of conjugates of $H$. The map is injective if and only if $H$ is self-normalizing in $G$, that is, if the normal closure of $H$ in $G$ is the whole of $G$.

The proof is part of the literature.

2.3. **The main lemma.** We shall prove the following:

**Lemma 3** (Permutation action on coset space)**.** Let $G$ be a group, $H$ a subgroup and $\sigma$ an automorphism of $G$. Consider the statements:

(1) $\sigma$ pushes forward to an inner automorphism for the permutation representation of $G$ on the coset space of $H$.
(2) $\sigma(K)$ and $K$ are conjugate subgroups for every conjugate $K$ of $H$

(1) $\implies$ (2) is always true. If $H$ is self-normalizing in $G$, then (2) $\implies$ (1).

*Proof.* Let $\rho$ be the permutation representation of $G$ on the coset space of $H$.

(1) $\implies$ (2): Let $\sigma'$ be an element of the symmetric group over $G/H$ that acts as a lift of $\sigma$. Let $K = gHg^{-1}$ be a conjugate of $H$. We shall show that $\sigma(K)$ is $sHs^{-1}$ where $sH = \sigma'(gH)$.

To prove this, observe that $K$ is precisely the stabilizer of the coset $gH$, and $sHs^{-1}$ is precisely the stabilizer of $sH$. Thus, it suffices to shw that $\sigma(K)$ is the stabilizer of $sH$.

Let $k \in K$. Write $k = ghg^{-1}$ for some $h \in H$.Then:

$$\begin{aligned}
\sigma(k).\sigma'(gH) &= (\sigma' \circ k \circ \sigma'^{-1}).(\sigma'(gH)) \\
\implies \sigma(k).\sigma'(gH) &= \sigma'.(k.(gH))) \\
\implies \sigma(k).\sigma'(gH) &= \sigma'.(ghg^{-1}gH)) \\
\implies \sigma(k).\sigma'(gH) &= \sigma'.(gH)) \\
\implies \sigma(k).sH &= sH
\end{aligned}$$

Thus, we have proved that $sH$ is invariant under $\sigma(K)$. The same argument also shows that $\sigma(K)$ is precisely the stabilizer of $sH$.

(2) $\implies$ (1): Suppose it is true that for every conjugate $K$ of $H$, $\sigma(K)$ is conjugate to $K$. Then $\sigma$ induces a permutation on the set of conjugates of $H$. Since $H$ is self-normalizing, we have, by the preceding lemma, a natural correspondence between the coset space of $H$ and the collection of conjugates of $H$. Thus, $\sigma$ gives rise to a permutation on the coset space of $H$. Call this permutation $\sigma'$.

We now need to prove that $\sigma'$ is indeed an extension of $\sigma$. In other words, we must show that $\sigma' \circ \rho(g) \circ \sigma'^{-1}$ is equivalent to $\rho(\sigma(g))$. That is, given any coset $lH$, we must show that the coset $\sigma(g)lH$ is the same as the coset $\sigma'(g\sigma'^{-1}(lH))$.

By the correspondence between cosets and stabilizers, it suffices to prove that the stabilizers of the cosets $\sigma(g)lH$ and $\sigma'(g\sigma'^{-1}(lH))$ are the same.

Since $\sigma'$ lifts $\sigma''$, the stabilizer of the coset $\sigma'^{-1}(lH)$ equals the coset $\sigma^{-1}(lHl^{-1})$. The stabilizer of $g\sigma'^{-1}(lH)$ is the coset $g\sigma^{-1}(lHl^{-1})g^{-1}$. Thus, the stabilizer of $\sigma'(g\sigma'^{-1}(lHl^{-1})g^{-1})$ is $\sigma(g)(lHl^{-1})\sigma(g)^{-1}$. This is the same as the stabilizer of $\sigma(g)lH$.

Hence Proved. □

2.4. **The main result.** We now supply the proof to the main result viz theorem 1.

**Lemma 4** (Permutation-extensible automorphisms are subgroup-conjugating)**.** Let $G$ be any group and $\sigma$ a permutation-extensible automorphism of $G$. Then $\sigma(H)$ is a conjugate of $H$ for any subgroup $H$ of $G$.

*Proof.* By lemma **??**, $\sigma$ has the property of being pushforwardable to an inner automorphism for any transitive permutation representation. Consider the permutation representation of $G$ on the coset space of $H$. This is a transitive permutation representation. Thus, $\sigma$ can be pushed forward to an inner automorphism for this representation. Now, applying lemma 3, we conclude that $\sigma(H)$ is conjugate to $H$. □

Combining lemmas 2 (which states that every extensible automorphism of a finite group is permutation-extensible) and 1 (which states that every permutation-extensible automorphism is subgroup-conjugating) we obtain a proof of the main theorem.

2.5. **A partial converse.**

**Lemma 5** (Automorphisms preserving a subgroup are pushforwardable)**.** Let $H$ be a subgroup of $G$. Then any automorphism of $G$ that preserves $H$ can be pushed forward to an inner automorphism for the permutation representation of $G$ on $H$.

*Proof.* Let $\sigma$ be an automorphism of $G$ that preserves $H$. Then $\sigma$ takes each left coset of $H$ to a left coset of $H$, and hence defines a permutation on the left coset space of $H$. Call this permutation $\sigma'$. We want to show that $\sigma'$ is a pushforward of $\sigma$. To prove this, it suffices to show that $\sigma' \circ \rho(g) \circ \sigma'^{-1} = \rho(\sigma(g))$ for any $g \in G$.

Consider a coset $lH$. Then $\sigma(g).lH$ is the coset $(\sigma(g)l)H$. The other side is $\sigma(g\sigma^{-1}(lH))$ which is the same as $\sigma(g)lH$. Thus, the two sides are the same and the proof is complete. □

Another definition:

**Definition** (Power automorphism)**.** An automorphism of a group is termed a **power automorphism**(defined) if it takes each element to within its subgroup.

From the above lemma, we have an easy corollary:

**Corollary 1.** Any power automorphism of a group is permutation-pushforwardable.

## 3. Small points I'll work on

**3.1. Subgroup-conjugating equals permutation-extensible.** I strongly suspect that any subgroup-conjugating automorphism is permutation-extensible. I am trying to modify the above proof to make the proof of this statement go through.

**3.2. Extensible implies permutation-extensible.** We have given the proof in the finite case, using the fact that the symmetric group on a finite set has no outer automorphisms (except the case where the finite set has exactly 6 elements). If the corresponding result holds for the symmetric group on an infinite set, then I would have shown that any extensible automorphism of any group is permutation-extensible.

Note that even as of now, the following is true:

**Lemma 6** (Extensible implies subgroup-conjugating for finite index)**.** Any extensible automorphism of a group must take any subgroup of finite index to a conjugate subgroup.

**3.3. Permutation-extensibility is preserved upon pushforwards.** I am omitting the proof of the following easy, but important observation:

**Observation 2** (Extensibility is preserved upon pushforwards)**.** If $\sigma$ is a permutation-extensible automorphism of a finite group $G$, and $N$ is a normal subgroup of $G$, then:

- $\sigma$ leaves $N$ invariant (because $\sigma$ is subgroup-conjugating)
- $\sigma$ induces a permutation-extensible automorphism on $G/N$.

The observation not only shows that permutation-extensible automorphisms are quotientable, it also shows that the property of being permutation-extensible *pushes forward.*

## 4. Combining permutation-extensibility with linear extensibility

**4.1. The combined result.** Combining the techniques of linear extensibility and permutation-extensibility, the following result comes out:

**Theorem 2** (Constraints on extensible automorphisms)**.** Let $G$ be a finite group and $\sigma$ an extensible automorphism of $G$. Then:

- $\sigma$ must map each element of $G$ to a conjugate element (that is, $\sigma$ is a class automorphism).
- $\sigma$ must map each subgroup of $G$ to a conjugate subgroup (that is, $\sigma$ is a subgroup-conjugating automorphism).

Clearly, any inner automorphism satisfies both these conditions. The converse question is: does any automorphism of a finite group satisfying both these conditions have to be inner? In other words, is the group of inner automorphisms precisely the intersection of the group of class automorphisms and the group of subgroup-conjugating automorphisms?

4.2. **Class automorphisms that are not inner: often not subgroup-conjugating.** The standard example of a class automorphism that is not inner, obtained by taking the semidirect product of the additive group of $\mathbb{Z}/8\mathbb{Z}$ with its multiplicative group and thne choosing a cocycle that is a *local* coboundary, is *not* a subgroup-conjugating automorphism. In fact, no example constructed in this manner can be subgroup-conjugating.

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