

LINEAR EXTENSIBILITY: A ROUTE TO EXTENSIBLE AUTOMORPHISMS (PART II)

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ABSTRACT. This is the second of a two-part article that describes the use of linear extensibility as a tool towards resolving the problems of characterizing extensible automorphisms, quotient-pullbackable automorphisms, and pushforwardable automorphisms. Complete proofs are supplied. The article also details inherent limitations of the linear extensibility approach.

1. THE PROBLEM WE WANT TO SOLVE

1.1. **Three big problems.** A quick recall of three definitions:

Definition. (1) An automorphism σ of a group G is termed **extensible**_(defined) if for any embedding $G \leq H$, there is an automorphism σ' of H such that $\sigma'|_G = \sigma$.

For a particular embedding $G \leq H$, the group of automorphisms of G that can be extended to automorphisms of H is denoted as **Extensible**(G ; H). The group of extensible automorphisms of G is thus given by:

$$\text{Extensible}(G) = \bigcap_{G \leq H} \text{Extensible}(G; H)$$

(2) An automorphism σ of a group G is termed **pushforwardable**_(defined) if for any homomorphism $\rho : G \rightarrow H$ there is an automorphism σ' of H such that $\rho \circ \sigma = \sigma' \circ \rho$.

For a particular homomorphism $\rho : G \rightarrow H$, the group of automorphisms of G that can be pushed forward via ρ is denoted as **Pushforwardable**(G ; ρ). The group of pushforwardable automorphisms is thus given by:

$$\text{Pushforwardable}(G) = \bigcap_{\rho: G \rightarrow H} \text{Pushforwardable}(G; \rho)$$

(3) An automorphism σ of a group G is termed **quotient-pullbackable**_(defined) if for any surjective homomorphism $\rho : H \rightarrow G$ there is an automorphism σ' of H such that $\rho \circ \sigma' = \sigma \circ \rho$.

For a surjective homomorphism $\rho : H \rightarrow G$, the group of automorphisms of G that can be *pulled back* to automorphisms of H is denoted as **Pullbackable**(G ; ρ). The group of quotient-pullbackable automorphisms of G is thus given by:

$$\text{Quot - Pullbackable}(G) = \bigcap_{\rho: H \rightarrow G} \text{Pullbackable}(G; \rho)$$

Some easy facts:

Observation 1. • Any inner automorphism of a group is both pushforwardable and quotient-pullbackable. In symbols:

$$\text{Inn}(G) \leq \text{Quot} - \text{Pullbackable}(G)$$

and

$$\text{Inn}(G) \leq \text{Pushforwardable}(G)$$

- Any pushforwardable automorphism is extensible.

$$\text{Pushforwardable}(G) \leq \text{Extensible}(G)$$

It is *conjectured* that for all groups, the properties of an automorphism being inner, extensible, pushforwardable, and quotient-pullbackable, are all equivalent.

Given a group, we can *measure the extent* to which this conjecture holds by asking the following questions:

- Points of Investigation 1.**
- What is the group of extensible automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\text{Inn}(G) = \text{Extensible}(G)$?
 - What is the group of pushforwardable automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\text{Inn}(G) = \text{Pushforwardable}(G)$?
 - What is the group of quotient-pullbackable automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\text{Inn}(G) = \text{Pushforwardable}(G)$?

1.2. The progress made in Part I. In Part I, we defined a notion of linear extensibility (linear pushforwardability) and used that to show that extensible automorphisms are class automorphisms, as well as to show that quotient-pullbackable automorphisms are class automorphisms. More precisely, we made two kinds of claims (for non-modular representations):

- Extensible automorphisms are linearly extensible (over a prime field)
- Linearly extensible automorphisms are class (over a sufficiently large field)

The inherent limitation of the approach is that *all* class automorphisms are linearly extensible, hence we cannot use the approach to put a stronger bound on extensible automorphisms.

In symbols, we proved that:

$$\text{Extensible}(G) \leq \text{Lin} - \text{extensible}(G; k) \leq \text{Class}(G)$$

and

$$\text{Quot} - \text{Pullbackable}(G) \leq \text{Lin} - \text{extensible}(G; k) \leq \text{Class}(G)$$

where k is a sufficiently large prime field. The limitation of the approach lies in the fact that $\text{Lin} - \text{extensible}(G; k) = \text{Class}(k)$ for a sufficiently large prime field, and hence we cannot show a tighter bound on $\text{Extensible}(G)$ using linear extensibility.

Here, I shall explore two variants:

- What are the linearly extensible automorphisms when the field is *not* sufficiently large? Note that this approach is not of use for obtaining stronger bounds on extensible automorphisms, but it has a nice answer.
- What are the linearly extensible automorphisms in case the characteristic of the field divides the order of the group?

2. GALOIS-CLASS AUTOMORPHISMS

2.1. Galois-conjugacy classes and local conjugacy classes.

Definition (Galois-conjugacy classes, local conjugacy classes). Let G be a group and k a field. Two elements of G are termed

- **Galois-conjugate**_(defined) if for any representation of G over k , the character takes the same value on the two elements.
- **locally conjugate**_(defined) if for any representation of G over k , the images of the elements are conjugate in the corresponding general linear group.

Conjugate \implies Locally conjugate over $k \implies$ Galois-conjugate over k

A result from representation theory characterizes the relation of being Galois-conjugate:

Fact 1 (Characterization of Galois-conjugacy classes). Let G be a group of exponent m , k a field whose characteristic does not divide the order of G , and K the splitting field for the m^{th} roots of unity over k .

The equivalence relation of being Galois-conjugate is the coarsest equivalence relation generated by the following two equivalences:

- The equivalence relation of being conjugate.
- The equivalence relation between an element and its r^{th} power where powering by r is in the Galois group K/k .

Moreover, the irreducible characters of G over k form an orthonormal basis for the space of functions on Galois-conjugacy classes.

Note that this result subsumes the result that characters span the space of class functions in the *sufficiently large* case.

2.2. Galois-class, local class and linearly pushforwardable. First, a definition:

Definition (Galois-class automorphism, local class automorphism). An automorphism of a group G with respect to a field k is termed a:

- **Galois-class automorphism**_(defined) if it takes each element to within its Galois-conjugacy class with respect to k .
- **local class automorphism**_(defined) if it takes each element to within its local conjugacy class with respect to k .

We are now in a position to state the main theorem:

Theorem 1 (Galois-class characterization of linearly extensible). Let G be a finite group, and k a field whose characteristic does not divide the order of G . Let $\sigma \in \text{Aut}(G)$. Then, the following are equivalent:

- (1) σ is a local class automorphism (that is, it preserves local conjugacy classes)
- (2) σ is a Galois-class automorphism (that is, it preserves Galois-conjugacy classes)
- (3) σ is linearly pushforwardable
- (4) σ is linearly extensible

It suffices to show that (1) \implies (2) \implies (3) \implies (4) \implies (1). The parts (1) \implies (2) and (3) \implies (4) are straightforward, so we shall show that (2) \implies (3) and (4) \implies (1).

2.3. Galois-class implies linearly pushforwardable.

Lemma 1 (Galois-class automorphisms are linearly pushforwardable). Let G be a finite group and k a field whose characteristic does not divide the order of G . Suppose σ is a Galois-class automorphism of G . Then for any linear representation $\rho : G \rightarrow GL_n(k)$ of G over k , σ pushes forward to an inner automorphism of $GL_n(k)$.

Proof. Consider the representations ρ and $\rho \circ \sigma$. For any g , $\sigma(g)$ is in the same Galois-conjugacy class as g , so they have the same character for every representation. Thus, the characters of ρ and $\rho \circ \sigma$ are identical. Hence, ρ and $\rho \circ \sigma$ are conjugate representations in $GL_n(k)$ and thus σ pushes forward to an inner automorphism in $GL_n(k)$. \square

2.4. Linearly extensible automorphisms are local class automorphisms.

Lemma 2 (Linearly extensible automorphisms are local class). Let σ be a linearly extensible automorphism of a group G over a field k . Then σ is a local class automorphism of G with respect to k .

Proof. Suppose σ is linearly extensible but not a local class automorphism. Then, there exists $g \in G$ such that g and $\sigma(g)$ are not in the same local conjugacy class. Hence, there exists a representation ρ such that $\rho(g)$ and $\rho(\sigma(g))$ are not conjugate in the general linear group. By the fact that any representation is a difference of two faithful representations and some basic facts about conjugacy properties of matrices, we can assume that there is a *faithful* representation ρ such that $\rho(g)$ and $\rho(\sigma(g))$ are not conjugate. Hence ρ and $\rho \circ \sigma$ are not conjugate representations. So, σ is not linearly extensible. \square

3. MODULAR CASE

3.1. Description of the modular case. The non-modular case is the case of representations of a group over a field whose characteristic does *not* divide the order of the group.

The other case, which we shall consider here, is where the characteristic of the field *does* divide the order of the group.

This has two subcases:

- The group is a p -group where p is the characteristic of the field. That is, there are no other prime divisor of the order of the group.
- The group has p and also has one or more other prime factor(s).

3.2. What carries over from the non-modular case? In the non-modular case, there were the following four important facts:

- (1) Any extensible automorphism is linearly extensible
- (2) Any linearly extensible automorphism is class (when the field is sufficiently large). In the general case, any linearly extensible automorphism is local class and hence also Galois-class.
- (3) Any class automorphism is linearly extensible
- (4) Any quotient-pullbackable automorphism is linearly extensible

Let us investigate each of these in the modular case.

3.3. Two fundamental differences. There are two fundamental differences between the modular and the non-modular case.

Firstly the *character* no longer determines the representation. To see just how drastic this can get, consider the case of representations of a p -group over the field F_p . We can prove that any such representation is *unipotent*, that is, it can be expressed with all the entries as upper triangular unipotent matrices. Thus, for any such representation, the character map is a *constant* function whose value equals the degree of the representation.

The second fundamental difference is that the vector space and the group acting on it no longer have relatively prime orders, hence the vector space *need not be characteristic* in the semidirect product.

3.4. Extensible automorphisms are linearly extensible. The result that extensible automorphisms are linearly extensible works even in the non-modular case, due to the second part of the mega-lemma proved in the previous article:

Lemma 3. Let $\rho : G \rightarrow GL_n(F_p)$ be a faithful representation of G over a prime field F_p . $A = F_p^n$ and $H = A \rtimes G$ via the representation. Then, if an automorphism σ of G is extensible to $GA_n(F_p) = A \rtimes GL_n(F_p)$ it can be pushed forward to an *inner* automorphism of $GL_n(F_p)$ (that is, it is *linearly pushforwardable* for the representation).

Proof. Since $A = F_p^n$ is characteristic in $GA_n(F_p)$, any lift of the automorphism of G to $GA_n(F_p)$ must leave A invariant. Since the automorphism already leaves G invariant, it restricts to an automorphism of $H = A \rtimes G$. Now, applying the Automorphism group action lemma, we obtain that the automorphism must be linearly extensible. \square

3.5. Is every class automorphism linearly extensible? To understand the drastic way in which modular and non-modular representation theory differ, consider the following definition:

Definition. Two representations $\rho_1, \rho_2 : G \rightarrow GL_n(k)$ are termed **locally conjugate**_(defined) if for any g , $\rho_1(g)$ and $\rho_2(g)$ are conjugate elements in $GL_n(k)$. A field is said to be **locally determining**_(defined) for a group if any two representations over the field that are locally conjugate are in fact conjugate.

Now a little claim:

Claim. When the characteristic of the field does not divide the order of the group, the field is locally determining for the group.

Proof. Since the two representations are locally conjugate, the character function associated with both representations is the same. From the fact that the character determines the representation (upto conjugacy) we conclude that the two representations are conjugate. \square

Further, the property of being “locally determining” is the crucial ingredient in showing that class automorphisms are always linearly pushforwardable:

Claim. If a field is locally determining for a group, then any local class automorphism of the group is linearly pushforwardable.

Proof. Let σ be any class automorphism of the group and ρ any representation. Clearly, ρ and $\rho \circ \sigma$ are *locally* conjugate, hence, by the “locally determining” fact, they are conjugate. Thus, σ pushes forward to an inner automorphism. \square

For the modular case, the character does *not* determine the representation. Can we still hope that the field is locally determining for the group? A counterexample was given by Professor Jonathan L. Alperin of Chicago University in email correspondence.

We begin with a little definition:

Definition. Given a field k , define the **unipotent representation**_(defined) of k as the representation that sends $a \in k$ to the matrix

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

I'll use the letter U for this representation and denote the matrix corresponding to a as $U(a)$.

Claim. Suppose a and b are two nonzero elements of a field k . Then:

$$\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

Thus, the unipotent matrices corresponding to a and b are conjugate.

Claim. The centralizer of any unipotent upper triangular non-identity matrix is precisely the collection of upper triangular matrices with equal diagonal entries. In particular, all unipotent upper triangular non-identity matrices have the same centralizer group.

From the above two claims, we get the following:

Claim. Let a, b and c be nonzero elements of the field k . Then, any matrix that conjugates $U(a)$ to $U(b)$ must conjugate $U(c)$ to $U(cb/a)$.

We can now explicitly construct the counterexample:

Claim. let k be a field (of dimension 2 or higher) and σ a group automorphism of k that is *not* a scalar multiplication. The the representations U and $U \circ \sigma$ are locally conjugate but not conjugate.

3.6. Is every linearly extensible automorphism class? The proof that any linearly extensible automorphism is a local class automorphism (lemma 2) does not depend on the characteristic of the field. Hence, every linearly extensible automorphism must preserve local conjugacy classes.

The question now is: what do local conjugacy classes look like?

Claim. Two elements in a p -group are in the same local conjugacy class with respect to an algebraically closed field of characteristic p if they generate conjugate subgroups.

Proof. It suffices to prove that if $g, h \in G$ generate the same subgroup, then for any representation over characteristic p , their images are conjugate.

We need to show that g and h have the same Jordan canonical form, or in other words, the same collection of Jordan blocks. Let r be such that $h = g^r$. Writing g in Jordan canonical form, and observe that all eigenvalues are 1. Raising to the r^{th} power, we obtain that the Jordan blocks of h refine those of g . By symmetry, the Jordan blocks of h refine those of g . Hence, g and h have the same Jordan decomposition and are thus conjugate. \square

Thus, the best result we can hope to squeeze from the local conjugacy class approach is that linearly extensible automorphisms are family automorphisms (that is, they take each element to an element generating a conjugate subgroup).

3.7. Is every quotient-pullbackable automorphism linearly extensible? To investigate the question of whether quotient-pullbackable automorphisms remain linearly extensible in the modular case, let's make a few definitions and recast the proof for the non-modular case:

Definition (Characteristic representation). A representation $\rho : G \rightarrow GL(V)$ is termed **characteristic**_(defined) if V is characteristic in the semidirect product $V \rtimes G$.

Note that in the previous article, we had shown that in the non-modular case, any representation is characteristic.

To study quotient-pullbackability in the modular case, we must concentrate only on characteristic representations. A couple of definitions and a lemma will help explain this:

Definition (Characteristic-linearly pushforwardable automorphism). An automorphism σ of a group G is termed **characteristic-linearly pushforwardable**_(defined) if it can be pushed forward to an inner automorphism for any characteristic representation over k .

Definition (Characteristic-local conjugacy classes). Given a group G and a field k , two elements of G are termed **characteristic-locally conjugate**_(defined) with respect to k if, for any characteristic representation of G over k , their images in the matrix group are conjugate elements.

The results connecting these ideas are:

Lemma 4 (Quotient-pullbackability and characteristic representations). Any quotient-pullbackable automorphism of a finite group G is characteristic-linearly pushforwardable over any field k . Hence, it also preserves characteristic-local conjugacy classes. That is:

Quotient-pullbackable \implies Characteristic-linearly pushforwardable \implies Characteristic-local class automorphism

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