

AUTOMORPHISM GROUPS AND EXTENSIBLE AUTOMORPHISMS (PART 1)

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ABSTRACT. The aim of this article is to consider the Extensible Automorphisms Problem for groups using their occurrence in “automorphism groups” We first try to show that for automorphism groups, extensible automorphisms are indeed inner. In Part 1, we outline possible directions of attack using the theory of automorphism groups of simple groups, that in one stroke generalizes linear extensibility and permutation extensibility.

1. THE PROBLEMS WE WANT TO SOLVE

1.1. **Three big problems.** A quick recall of three definitions:

Definition. (1) An automorphism σ of a group G is termed **extensible**_(defined) if for any embedding $G \leq H$, there is an automorphism σ' of H such that $\sigma'|_G = \sigma$.

For a particular embedding $G \leq H$, the group of automorphisms of G that can be extended to automorphisms of H is denoted as **Extensible**(G ; H). The group of extensible automorphisms of G is thus given by:

$$\text{Extensible}(G) = \bigcap_{G \leq H} \text{Extensible}(G; H)$$

(2) An automorphism σ of a group G is termed **pushforwardable**_(defined) if for any homomorphism $\rho : G \rightarrow H$ there is an automorphism σ' of H such that $\rho \circ \sigma = \sigma' \circ \rho$.

For a particular homomorphism $\rho : G \rightarrow H$, the group of automorphisms of G that can be pushed forward via ρ is denoted as **Pushforwardable**(G ; ρ). The group of pushforwardable automorphisms is thus given by:

$$\text{Pushforwardable}(G) = \bigcap_{\rho: G \rightarrow H} \text{Pushforwardable}(G; \rho)$$

(3) An automorphism σ of a group G is termed **quotient-pullbackable**_(defined) if for any surjective homomorphism $\rho : H \rightarrow G$ there is an automorphism σ' of H such that $\rho \circ \sigma' = \sigma \circ \rho$.

For a surjective homomorphism $\rho : H \rightarrow G$, the group of automorphisms of G that can be *pulled back* to automorphisms of H is denoted as **Pullbackable**(G ; ρ). The group of quotient-pullbackable automorphisms of G is thus given by:

$$\text{Quot - Pullbackable}(G) = \bigcap_{\rho: H \rightarrow G} \text{Pullbackable}(G; \rho)$$

Some easy facts:

Observation 1. • Any inner automorphism of a group is both pushforwardable and quotient-pullbackable. In symbols:

$$\text{Inn}(G) \leq \text{Quot} - \text{Pullbackable}(G)$$

and

$$\text{Inn}(G) \leq \text{Pushforwardable}(G)$$

- Any pushforwardable automorphism is extensible.

$$\text{Pushforwardable}(G) \leq \text{Extensible}(G)$$

It is *conjectured* that for all groups, the properties of an automorphism being inner, extensible, pushforwardable, and quotient-pullbackable, are all equivalent.

Given a group, we can *measure the extent* to which this conjecture holds by asking the following questions:

- Points of Investigation 1.** • What is the group of extensible automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\text{Inn}(G) = \text{Extensible}(G)$?
- What is the group of pushforwardable automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\text{Inn}(G) = \text{Pushforwardable}(G)$?
 - What is the group of quotient-pullbackable automorphisms of the group? Is it exactly the same as the group of inner automorphisms? That is, is $\text{Inn}(G) = \text{Pushforwardable}(G)$?

1.2. **The results we want to show.** In this article, we concentrate on the problem of extensible automorphisms only, relegating the problem of quotient-pullbackable automorphisms to the background. We shall proceed roughly as follows:

- (1) For automorphism groups of certain kinds, every extensible automorphism is inner.
- (2) Assuming a conjecture, every extensible automorphism of any automorphism group is inner.
- (3) For certain kinds of subgroups of automorphism groups, every iteratively extensible automorphism is inner.

2. THE DEFINITIONAL GROUNDWORK AND PROOF

2.1. **Automorphism group action lemma.** This automorphism group action lemma is a *mild* generalization of the automorphism group action lemma that we encountered in the article on linearly extensible automorphisms. The proof is the same.

Lemma 1 (Automorphism group action lemma). Let $\rho : G \rightarrow \text{Aut}(N)$ be a homomorphism. Let M be a group containing copies of N and G such that $NG = M$, $N \trianglelefteq M$ and the action of G on N by conjugation is precisely ρ . Then, any automorphism of G that lifts to an N -preserving automorphism of M must push forward to an inner automorphism of $\text{Aut}(N)$ via ρ .

Proof. Suppose σ is an automorphism of G that is semidirectly extensible to M . This means that there is an automorphism σ' on M such that σ' leaves N invariant and its action on the subgroup G is the same as σ .

Since the restriction of σ' to N is an automorphism of N , the restriction is an element of $\mathbf{Aut}(N)$. Let's call this element a .

Before proceeding, let's just straighten the notation:

- σ' is an automorphism of M
- The restriction of σ' to N is $a \in \mathbf{Aut}(N)$.
- The restriction of σ' to G is $\sigma \in \mathbf{Aut}(G)$.

We claim that:

$$\rho \circ \sigma = c_a \circ \rho$$

where c_a denotes conjugation by a . This shows that σ can be pushed forward to an inner automorphism of $\mathbf{Aut}(N)$.

To prove this, it suffices to show that for any $g \in G$, and $n \in N$, $((\rho \circ \sigma)(g)).n = ((c_a \circ \rho)(g)).n$. Assume $n = a.m$ and expand the left-hand-side:

$$\begin{aligned} & ((\rho \circ \sigma)(g)).(n) = \sigma(g).(n) \\ \implies & ((\rho \circ \sigma)(g)).(n) = \sigma'(g).(n) \text{ because } \sigma = \sigma'|_G \\ & ((\rho \circ \sigma)(g)).(n) = \sigma'(g).(a.m) \text{ because } n = a.m \\ \implies & ((\rho \circ \sigma)(g)).(n) = \sigma'(g)(a.m)\sigma'(g^{-1}) \text{ because } \sigma'(g) \text{ acts by conjugation} \\ \implies & ((\rho \circ \sigma)(g)).(n) = \sigma'(g)\sigma'(m)\sigma'(g^{-1}) \text{ because } a = \sigma'|_N \\ \implies & ((\rho \circ \sigma)(g)).(n) = \sigma'(gmg^{-1}) \text{ because } \sigma' \text{ is an automorphism} \\ \implies & ((\rho \circ \sigma)(g)).(n) = a.(gmg^{-1}) \text{ because } a = \sigma'|_N \\ \implies & ((\rho \circ \sigma)(g)).(n) = a.\rho(g).m \text{ because } g \text{ acts by conjugation} \\ \implies & ((\rho \circ \sigma)(g)).(n) = a.\rho(g).a^{-1}.n \text{ because } n = a.m \\ \implies & ((\rho \circ \sigma)(g)).(n) = (a\rho(g)a^{-1}).n \\ \implies & ((\rho \circ \sigma)(g)).(n) = ((c_a \circ \rho)(g)).n \end{aligned}$$

Hence proved. □

Particular corollaries of interest:

Corollary 1 (Characteristic situation). Let $\rho : G \rightarrow \mathbf{Aut}(N)$ be a homomorphism, and M a group containing both N and G such that N is characteristic in M , $NG = M$, and the action of G on N by conjugation is precisely ρ . Then, any automorphism of G that extends to M must push forward to an inner automorphism of $\mathbf{Aut}(N)$ via ρ .

Proof. Since N is characteristic in M , any automorphism of M must leave N invariant. Hence, any extension of an automorphism of G to an automorphism of M must also restrict to an automorphism of N . □

2.2. The potentially characteristic weakening. First, a definition:

Definition (Potentially characteristic subgroup). A subgroup H of a group G is termed **potentially characteristic**_(defined) in G if there exists a group K containing G such that H is characteristic in K .

It turns out that the requirement of being characteristic can be weakened to the requirement of being potentially characteristic:

Lemma 2 (Potentially characteristic situation). Let $\rho : G \rightarrow \text{Aut}(N)$ be a homomorphism, and M be a group containing both N and G such that N is potentially characteristic in M , and the action of G on N by conjugation is precisely ρ . Let K be a group containing M such that N is characteristic in K . Then, any automorphism of G that extends to an automorphism of K must push forward to an *inner* automorphism of $\text{Aut}(N)$.

Proof. Any automorphism of K obtained by extending an automorphism of G must leave N invariant, and since it already leaves G invariant, it restricts to an automorphism of M . We can now apply the Automorphism group action lemma. \square

2.3. The NPC conjecture. The NPC conjecture states the following:

Conjecture 1 (Normal subgroups are potentially characteristic (NPC)). Let H be a normal subgroup of a group G . Then, there is a group K containing G such that H is characteristic in K .

2.4. Some quick corollaries for automorphism groups. A few lemmas:

Lemma 3 (Vector space characteristic in its affine group). Let V be a vector space over any field, and $GL(V)$ its group of linear automorphisms. Let $GA(V)$ denote the semidirect product. Then V is a characteristic subgroup inside $GA(V)$.

Proof. We first make two observations:

- V is a minimal normal subgroup of $GA(V)$: First, observe that since $GL(V)$ acts transitively on the nonzero elements of V , V does not have a proper nontrivial $GL(V)$ -invariant subspace, and hence V is a minimal normal subgroup of $GA(V)$.
- V is self-centralizing in $GA(V)$: Any element outside V acts on V nontrivially by conjugation. Thus, V is self-centralizing in $GA(V)$.

Suppose V is not characteristic. Then, there exists σ , an automorphism of $GA(V)$ such that $\sigma(V) \neq V$. Then, since V is minimal normal, $\sigma(V)$ must also be minimal normal, hence they must intersect trivially. V and $\sigma(V)$ are thus trivially intersecting normal subgroups, and hence every element in V commutes with every element in $\sigma(V)$. But this forces $\sigma(V) = V$, a contradiction. \square

Lemma 4 (Characteristically simple non-Abelian group centerless, characteristic in automorphism group). Let N be a characteristically simple non-Abelian group. Then N is centerless and is characteristic in $\text{Aut}(N)$ via the natural embedding.

Proof. Since N is characteristically simple, its center must either be the whole of N or the trivial subgroup. Since N is non-Abelian, the center cannot be the whole of N , hence the center is the trivial subgroup. Thus, N is centerless.

Since N is characteristically simple, it has no proper automorphism-invariant subgroups. Thus, there is proper nontrivial subgroup of N that is normal in $\text{Aut}(N)$, hence N is a minimal normal subgroup of $\text{Aut}(N)$.

Further, no nontrivial element of $\text{Aut}(N)$ commutes with every element of N because every nontrivial automorphism acts nontrivially on N by conjugation.

Suppose N is not characteristic. Then, there exists an automorphism σ of $\text{Aut}(N)$ such that $\sigma(N) \neq N$. Since N is minimal normal, so is $\sigma(N)$. Their intersection is again a normal subgroup, hence they intersect trivially. So, every element in N commutes with every element in $\sigma(N)$, contradicting the fact that the centralizer of N is trivial.

Thus, N is characteristic in $\text{Aut}(N)$. □

- Theorem 1** (Mega-theorem on automorphism groups). (1) Let N be a group that is characteristic in $N \rtimes \text{Aut}(N)$. Then, any extensible automorphism of $\text{Aut}(N)$ is inner. In particular, if N is an elementary Abelian group, any extensible automorphism of N is inner.
- (2) Let N be a centerless group that is characteristic in $\text{Aut}(N)$ by the natural embedding. Then, any automorphism of $\text{Aut}(N)$ is inner. In particular, if N is a characteristically simple non-Abelian group, then any automorphism of $\text{Aut}(N)$ is inner.
- (3) If the NPC conjecture holds, then for any group N , every extensible automorphism of $\text{Aut}(N)$ is inner.

Proof. Proof of (1): Set $G = \text{Aut}(N)$, $M = N \rtimes G$. Now apply the automorphism group action lemma.

The particular case follows from lemma 3.

Proof of (2): Set $G = M = \text{Aut}(N)$. Then apply the automorphism group action lemma. Observe here that since $G = M$, the *extension* to M of an automorphism of G is the *same* automorphism.

The particular case follows from lemma 4.

Proof of (3): Set $G = \text{Aut}(N)$, $M = N \rtimes G$. If NPC holds, then N is potentially characteristic in M . Apply lemma 2 to obtain the result. □

3. THEORY OF WC SUBGROUPS

3.1. **Certain subgroups of automorphism groups.** A little definition first:

Definition (WC subgroup). A subgroup H of a group G is termed a **WC subgroup**_(defined) if the following equivalent conditions hold:

- $HC_G(H) = N_G(H)$

- Any inner automorphism of G that leaves H invariant, must restrict to an *inner* automorphism of H .

Definition (sub-WC subgroup). Let n be a positive integer. A subgroup H is termed a **sub-WC subgroup**_(defined) of depth n if there is an ascending chain of subgroups:

$$H = H_0 \leq H_1 \leq \dots \leq H_n = G$$

where each H_i is a WC subgroup of H_{i+1} .

3.2. The extensibility operator and WC subgroups. One natural direction to generalize the notion of extensible automorphism is using the extensibility operator.

Definition (Extensibility operator). The extensibility operator is a map that takes an input a property of group automorphisms and gives as output a property of group automorphisms. The result of applying the extensibility operator to a property p of group automorphisms is denoted as $E(p)$. $E(p)$ is defined in terms of p as follows:

An automorphism σ of a group G has property $E(p)$ if for any embedding $G \leq H$ of groups, there is an automorphism σ' of H such that $\sigma'|_G = \sigma$ and σ' satisfies the property p as an automorphism of H .

Now an important result connecting the extensibility operator and WC subgroups:

Lemma 5 (WC subgroups and the extensibility operator). Let p be a property of groups and G a group such that any automorphism of G satisfying property p is inner. Then, if H is a WC subgroup of G , any automorphism of H satisfying $E(p)$ is inner.

Proof. Suppose σ is an automorphism of H satisfying $E(p)$. Then, there exists an automorphism σ' of G satisfying property p such that $\sigma'|_H = \sigma$. Since every automorphism of G satisfying property p is inner, σ' is an inner automorphism of G . Also, σ' restricts to an automorphism of H . Since H is a WC subgroup of G , the restriction of σ' to H must be an inner automorphism of H , hence σ is inner. \square

3.3. Iteratively extensible automorphisms. We had earlier defined the notion of extensible automorphism. We now define a notion of l -extensible automorphisms for nonnegative integers l :

Definition (Iteratively extensible automorphism). We define, by induction on l , the notion of l -extensible automorphism.

- Every automorphism of a group is 0-extensible
- The property of being $(l + 1)$ -extensible is obtained by applying the extensibility operator to the property of being l -extensible.

Now an easy lemma using the previous lemma on the extensibility operator:

Lemma 6 (Iteratively extensible and sub-WC). Let H be a sub-WC subgroup of depth l in a group G . If every k -extensible automorphism of G is inner, then every $(k + l)$ -extensible automorphism of H is inner.

Proof. Since H has depth l , there is an ascending chain:

$$H = H_0 \leq H_1 \leq \dots \leq H_l = G$$

where each H_i is a WC subgroup in H_{i+1} .

Note first that H_l has the property that every k -extensible automorphism is inner. For $0 \leq m \leq l - 1$, the following holds as a consequence of lemma 5:

If every $k + l - m - 1$ -extensible automorphism of H_{m+1} is inner, then every $k + l - m$ -extensible automorphism of H_m is inner.

Applying this repeatedly, we obtain, from the starting fact that every k -extensible automorphism of H_l is inner, the fact that every $(k + l)$ -extensible automorphism of H_0 is inner. \square

3.4. Recasting an earlier result.

Theorem 2 (Extensible automorphisms of WC subgroups). Let G be a group with an embedding as a WC subgroup of an automorphism group $\mathbf{Aut}(N)$. Then:

- (1) If N is a centerless group that is characteristic in its semidirect product, then every extensible automorphism of G is inner.
- (2) If N is an elementary Abelian group, then every extensible automorphism of G is inner
- (3) If the NPC conjecture holds, then regardless of what N is, every extensible automorphism of G is inner.

Proof. Proof of (1) follows from the fact that every automorphism of $\mathbf{Aut}(N)$ is inner, and lemma 5.

Proof of (2) arises as follows: Consider the semidirect product $M = N \rtimes G$. N is potentially characteristic in M because N is characteristic in $K = N \rtimes \mathbf{Aut}(N)$. Thus, applying lemma 2 we obtain that any automorphism σ of G that extends to K must push forward to an inner automorphism of $\mathbf{Aut}(N)$. Since the mapping from G to $\mathbf{Aut}(N)$ is injective, this is the same as requiring that σ extend to an inner automorphism (say σ') of $\mathbf{Aut}(N)$. Since G is a WC subgroup of $\mathbf{Aut}(N)$, and σ' restricts to G , the restriction to G of σ' must be an inner automorphism of G . Hence, σ is an inner automorphism of G .

Since any extensible automorphism of G must extend to K , we conclude that any extensible automorphism of G is inner.

Proof of (3) arises as follows: Let $M = N \rtimes G$. Assuming the NPC, N is potentially characteristic in M . That is, there exists a group K containing M such that N is characteristic in K . Let σ be an automorphism of G that extends to K . Then by lemma 2, σ must push forward to an inner automorphism of $\mathbf{Aut}(N)$. Since G is a WC subgroup of $\mathbf{Aut}(N)$, and σ' restricts to G , the restriction to G of σ' must be an inner automorphism of G . Hence, σ is an inner automorphism of G .

Since any extensible automorphism of G must extend to K , we conclude that any extensible automorphism of G is inner. \square

4. FOR WHAT GROUPS IS THE PROBLEM SETTLED?

4.1. EAC true groups.

Definition (EAC true group). Let n be a nonnegative integer. A group G is termed **n -EAC true**_(defined) if every n -extensible automorphism of G is inner.

- A group is 0-EAC true if every automorphism of it is inner. Such a group is also termed **quasi-complete**_(defined).
- A group is 1-EAC true if every extensible automorphism of it is inner. The 1 can be dropped and such a group is termed **EAC true**_(defined).
- A group is ω -**EAC true**_(defined) if every automorphism of the group that is n -extensible for every positive integer n , is inner.

Previous results can be reformulated in the language of EAC trueness:

- Observation 2.**
- The automorphism group of any elementary Abelian group is 1-EAC true. This follows from theorem 1, part (1).
 - The automorphism group of any WC subgroup of an elementary Abelian group is 1-EAC true. This again follows from theorem 1, part (1).
 - The automorphism group of any centerless group that is characteristic in its automorphism group, is 0-EAC true. This follows from theorem 1, part (2).
 - If the NPC conjecture holds, then any WC subgroup of an automorphism group is 1-EAC true. This follows from theorem 1, part (3).
 - Any l -sub-WC subgroup of a k -EAC true group is $(k + l)$ -EAC true. This follows from lemma 6.

4.2. Simulating linear representations in greater generality. Linear representation theory has been used to show that every extensible automorphism of a group preserves conjugacy classes. The idea was to look at a number of different linear representations over a field in such a way that the only automorphisms that push forward in all cases are those that preserve conjugacy classes (it turns out later that we can find a single representation that does the job).

The idea is thus the following construction problem:

Given a group G , construct a family of collections (N_i, M_i, ρ_i) where N_i is a group, ρ_i is a map $G \rightarrow \text{Aut}(N_i)$, and $M_i = GN_i$ is a group in which N_i is potentially characteristic and the action of G on N_i by conjugation is precisely ρ_i . If we can show that any automorphism of G that pushes forward to an inner automorphism for each ρ_i must be inner, then we have shown that G is 1-EAC true.

5. THE CS EXTENSIBILITY APPROACH

5.1. Characteristically simple extensibility. To parallel the notion of linear extensibility, we have a notion of CS extensibility:

Definition (CS pushforwardable, CS extensible automorphism). An automorphism σ of a group G is termed **CS pushforwardable**_(defined) if it can always be pushed forward to an *inner* automorphism for any map $\rho : G \rightarrow \mathbf{Aut}(N)$ where N is a characteristically simple group.

An automorphism σ of a group G is termed **CS extensible**_(defined) if it can always be lifted to an *inner* automorphism for any injective homomorphism $\rho : G \rightarrow \mathbf{Aut}(N)$ where N is a characteristically simple group.

Lemma 7 (Pushforwardable automorphisms are CS pushforwardable). Any extensible automorphism σ of a group G is CS extensible.

Proof. Let N be a characteristically simple group and $\rho : G \rightarrow \mathbf{Aut}(N)$ be a map. We need to show that if σ is an extensible automorphism of G , then σ extends to an inner automorphism of $\mathbf{Aut}(N)$.

We consider two cases:

- N is Abelian: In this case, N is an elementary Abelian group. Take $K = N \rtimes \mathbf{Aut}(N)$. N is characteristic in the semidirect product (because of lemma 3). Thus, if σ' is a lift of σ to K , then σ' restricts to N as well as to G . Thus σ' restricts to an automorphism of $M = N \rtimes G$, and we can apply lemma 1.
- N is non-Abelian: Then, N is centerless and characteristic in its automorphism group. Thus, any automorphism of $\mathbf{Aut}(N)$ is inner. Since σ is an extensible automorphism of G , it must extend to an inner automorphism of $\mathbf{Aut}(N)$

□

CS extensibility is an important notion because it subsumes two kinds of extensibility we have seen:

- Lemma 8** (CS extensible implies linearly, permutation extensible). (1) Any CS extensible automorphism is linearly extensible over a prime field.
(2) Any CS extensible automorphism is a permutation-extensible automorphism.

Proof. Proof of (1): Let F_p be a prime field and σ a CS extensible automorphism of a group G . We need to show that if N is a vector space over F_p , and ρ a faithful linear representation of G via ρ , then σ can be lifted via ρ .

Note first that a linear representation $G \rightarrow GL(N)$ is the same as an injective map $G \rightarrow \mathbf{Aut}(N)$, because the group-theoretic automorphisms of N are the same as its vector space automorphisms (this is where we use that N is a vector space over a prime field). Further, N is a characteristically simple group, and σ is CS extensible. Hence, σ can be extended via ρ , and we are done.

Proof of (2): We use the fact that S_n is itself the automorphism group of a simple group (namely A_n for sufficiently large n). □

5.2. What do we know about CS extensible automorphisms? We know the following:

- Any CS extensible automorphism of a group is a class automorphism, that is, it takes each element to within its conjugacy class.

- Any CS extensible automorphism of a group is a subgroup-conjugating automorphism, that is, it takes each subgroup to a conjugate subgroup.

6. SUMMARY AND POINTS OF FURTHER EXPLORATION

6.1. **Points raised by the article.** The article raises the following questions and possibilities:

- (1) **The NPC conjecture:** This states that any normal subgroup is potentially characteristic. If we are somehow able to establish the NPC conjecture, then any WC subgroup of an automorphism group is EAC true.
- (2) **sub-WC subgroups:** Any group that occurs as a sub-WC subgroup of a special kind of automorphism group satisfies the property that its k -extensible automorphisms are inner (where k is the depth of the sub-WC subgroup). If the NPC conjecture were also established, then the corresponding statement holds for all sub-WC subgroups.
- (3) **System of representations:** Just as, in the linear case, we sought to consider a family of representations that would separate conjugacy classes, we can, in the general case, consider a family of group actions so that the only automorphisms that extend for all the corresponding group actions are the inner ones.
- (4) **CS extensible automorphisms:** Both linear extensibility and permutation extensibility arise as special cases of a general notion of CS extensibility. It might thus make sense to concentrate on showing that all CS extensible automorphisms of a group are inner.

6.2. **What we shall explore in Part 2.** In Part 2, we shall look at a list of all the simple groups and explore, for each of them, the question: what are the automorphisms that are CS extensible for characteristically simple groups that arise as powers of these simple groups? Since we know that any characteristically simple group is indeed a power of a simple group, this case-by-case analysis will comprehensively cover the question of CS extensibility.

We shall also look at a corresponding *representation theory* over simple groups, analogous to linear representation theory.

Finally, we shall try to obtain a complete listing of groups that occur as WC subgroups of automorphism groups of characteristically simple groups.

INDEX

- ω -EAC true group, 8
- n -EAC true group, 8
- automorphism
 - CS extensible, 9
 - CS pushforwardable, 9
 - extensible, 1
 - pushforwardable, 1
 - quotient-pullbackable, 1
- CS extensible automorphism, 9
- CS pushforwardable automorphism, 9
- EAC true group, 8
- extensible automorphism, 1
- group
 - ω -EAC true, 8
 - n -EAC true, 8
 - EAC true, 8
 - quasi-complete, 8
- potentially characteristic subgroup, 4
- pushforwardable automorphism, 1
- quasi-complete group, 8
- quotient-pullbackable automorphism, 1
- sub-WC subgroup, 6
- subgroup
 - potentially characteristic, 4
 - sub-WC, 6
 - WC, 5
- WC subgroup, 5