EXTENSIBLE AUTOMORPHISMS: A POSSIBLE APPROACH

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Abstract. It is easy to see that an inner automorphism of a group can always be extended to an inner automorphism of any group containing it. Are inner automorphisms the only such automorphisms? This is the problem of extensible automorphisms. Here, I discuss the problem of extensible automorphisms and its generalizations, as well as the progress that has been made on the problem via correspondence between Dr. Isaacs and me.

1. The problem statement

1.1. A special property of inner automorphisms.

Before beginning with the main content of this article, we make an important observation about **inner automorphisms**.

An inner automorphism of a group is a map of the form $x \mapsto gxg^{-1}$ where g is a fixed element of the group. The inner automorphism corresponding to g is also called the **conjugation** by g or the **transform** by g.

An automorphism on a group is said to **lift** or **extend** an automorphism on a subgroup if its restriction to the subgroup is that automorphism on the subgroup.

Given a group, an **embedding** of that into another group is an identification of the given group as a subgroup of the other group.

Any inner automorphism of a subgroup can naturally be lifted to an inner automorphism of the group. For, suppose the inner automorphism is obtained by conjugation by a particular element in the subgroup. Then conjugation by that same element also defines an inner automorphism of the group, that lifts the inner automorphism on the subgroup.

Another way of putting the above statement is that inner automorphisms lift for any embedding.

The above method lifts an inner automorphism to an *inner* automorphism. Thus, the automorphism obtained upon extension can again be extended. Moreover, if $H \leq K \leq G$ are groups then extending first from H to K and then from K to G gives the same result as directly extending from H to G. This is expressed by saying that inner automorphisms can be extended in a **commuting** fashion.[1]

All these observations raise the following questions:

- Why are inner automorphisms so nice?
- What are the consequences of their being so nice?
- Are there other automorphisms that have some of these properties?

These questions (in particular the last among them) have spurred on the problem of **extensible automorphisms** that is the subject of this article.

The author would like to thank Dr. Isaacs for his suggestions and proofs that have led to the partial solution obtained so far.

[1]There is a subtlety here. The inner automorphism could be induced by conjugation by more than one element, and as such, there is no natural way of deciding which element to use when extending. Once the element has been fixed, however, the manner of extending is natural.

1.2. **The original problem statement.** The general problem statement is:

Problem 1 (Which automorphisms are extensible?)**.** *An* **extensible automorphism** *(*my own terminology*) of a group is defined as an automorphism that can be lifted to an automorphism for any embedding into another group. Which automorphisms of a group are extensible?*

The following basic facts are direct:

- Every **inner automorphism** (that is, conjugation by some group element) is extensible. Thus, the property of being an inner automorphism is stronger than the property of being an extensible automorphism.
- The property of extensibility is closed under composition and inversion. Thus, the set of extensible automorphisms forms a subgroup of the group of all automorphisms.

It seems, from examination of elementary cases, that extensible automorphisms are precisely the inner automorphisms. This is my conjecture.

1.3. **Multiple extensions.** For convenience, we use the notation:

$$\alpha \leftarrow \beta$$

where α and β are both automorphism properties, to indicate a subgroup property where every automorphism satisfying property β in the subgroup can be lifted to an automorphism satisfying α in the whole group. Then, it is true that:

$$\text{Inner automorphism} \leftarrow \text{Inner automorphism}$$

is held for every subgroup, that is, the above property is the tautology.

On the other hand, the property:

$$\text{Inner automorphism} \leftarrow \text{Automorphism}$$

is equivalent to the subgroup being **fully normalized**(*infrequently used terminology*): the Weyl group of the subgroup[2] is the whole automorphism group.

The property:

$$\text{Automorphism} \leftarrow \text{Automorphism}$$

is not true for all subgroups. This property of subgroups is called the **automorphism extension property**. (*my own terminology*)The trivial subgroup and the group itself satisfy the automorphism extension property, but there are many subgroups which do not. For instance, there are automorphisms of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ that do not lift to automorphisms of the group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$ (in which it is embedded). Note that all fully normalized subgroups have the automorphism extension property.

On the other hand, if the property on the right is strengthened somewhat, we get:

$$\text{Automorphism} \leftarrow \text{Extensible automorphism}$$

This is true for every subgroup. This is because, by definition, an extensible automorphism is one that lifts to an automorphism for every embedding.

[2] The Weyl group of a subgroup is the quotient of its normalizer (in the group) by its centralizer (in the group), embedded naturally in the automorphism group of the subgroup. In geometric contexts, it often arises for self centralizing subgroups (such as tori) in which case it becomes the quotient of the normalizer of the subgroup

Extensible automorphisms form the weakest class that satisfy this property. That is, the property on the right cannot be weakened further. Extensible automorphisms comprise *all* the automorphisms that lift to automorphisms of the whole group.

This leads to another question: what are those automorphisms that can always be lifted to *extensible automorphisms*? It is not clear that all extensible automorphisms can be lifted to extensible automorphisms. We shall call an automorphism a "2 extensible automorphism" if it lifts to an extensible automorphism. This leads us to the next problem:

Problem 2 (Multiply extensible automorphisms)**.** *Suppose every automorphism of a group is* 0 *extensible. Call an automorphism a "*"k **extensible automorphism"** *(*my own terminology*) if for every embedding in another group, it can be lifted to a* $(k-1)$ *extensible automorphism of that group. Call an automorphism an* **"ω extensible automorphism"** *(*my own terminology*) if it is k extensible for all k. What are the k extensible automorphisms and ω extensible automorphisms?*

Once again, we have the following observations:

- Inner automorphisms are ω extensible. More generally, if α is an automorphism property such that $\alpha \leftarrow \alpha$ is tautologically true, then any automorphism satisfying α is ω extensible.
- The set of k extensible automorphisms, for any k, forms a subgroup of the automorphism group. Moreover, these subgroups form a descending chain as k increases, and their intersection is the subgroup of ω extensible automorphisms.
- If a subgroup is invariant under all k extensible automorphisms of the group, then it is invariant under all $(k+1)$ extensible automorphisms of any intermediate subgroup. Thus, in particular, characteristic subgroups are invariant under all extensible automorphisms of the group.

Inner automorphisms are in some sense even better than ω extensible automorphisms, because for inner automorphisms, the manner of extending is canonical, and so if we extend using two different intermediate chains, the final automorphism obtained is the same.

Of course, if the original conjecture, that all extensible automorphisms are inner comes true, then all the above properties (k extensibility for $k \geq 1$, ω extensibility) will become equivalent.

1.4. **A stronger formulation.** An important fact relating automorphisms and inner automorphisms is :

> Given a group, there is a group containing it, such that every automorphism of the group lifts to an inner automorphism of the bigger group (such an embedding is termed **fully normalized**). For instance, we can take the bigger group to be the semidirect product of the group with its automorphism group (this product is termed the **holomorph**).

Thus, by choosing a good embedding, we can make every automorphism inner.

In particular, this implies the somewhat weaker statement:

> Given a group and an automorphism of it, there is a bigger group containing the group so that the automorphism extends to an inner automorphism of the bigger group.

Note that this is weaker because here we do not assume that a *single group* suffices for all automorphisms.

The problem of showing that inner automorphisms are in fact the only extensible automorphism requires us to do something of the reverse nature – find a group where the *only* automorphisms that extend are the inner ones. This leads to the following problem formulation:

Problem 3. *Given a group, can we embed it in a group such that the only automorphisms of the smaller group that lift to automorphisms of the bigger group are inner?*

For Abelian groups, this translates to saying that "no nontrivial automorphism lifts to an automorphism of the bigger group". Note that *a priori*, a positive answer to this statement is stronger than the assertion that "every extensible automorphism is inner". In the above statement, we look for a *single group* so that the embedding in that group can be used to prove that every extensible automorphism is inner.

The above problem is definitely a more concrete one to work with. We need to develop tools that will help us solve the above problem constructively.

1.5. **A somewhat varied formulation.** A slight variant of the problem is:

Problem 4. *Given a group, can we embed it in a group such that given any two elements in the smaller group that are not conjugate in it, there is no automorphism of the bigger group taking one to the other?*

If the group we start with is Abelian, then this asks for an embedding in a group such that no automorphism of the bigger group takes one element of the smaller group to another element in the smaller group.

This differs from the previous problem formulation in some important respects. A positive answer to this problem will indicate that the only extensible automorphisms of a group are those that preserve conjugacy classes. However, it will not succeed in showing that these are inner automorphisms, because there are outer automorphisms that preserve conjugacy classes.

On the other hand, a positive answer to problem 3 will not directly indicate a positive answer to this, because here, the automorphism of the bigger group in question need not preserve the smaller group in its entirety.

However, there is one version that can be thought of as the "strongest of all" because it will imply all the previous ones:

Problem 5. *Given a group, can we embed it in a group such that the following condition holds: "if an isomorphism between two subgroups of the smaller group can be lifted to an automorphism of the bigger group, then it can be lifted to an inner automorphism of the smaller group"?*

If the group with which we start is Abelian, then the above condition will imply that there is an embedding such that "the only isomorphisms of subgroups that lift to automorphisms of the bigger group are identity isomorphisms".

In the next section, we shall propose a possible way of constructing such a group.

1.6. **Where we are now.** So far, we have seen that:

- Extensible automorphisms are those automorphisms that lift to automorphisms for any embedding. These can also be called 1 extensible automorphisms. We can inductively define k extensible automorphisms and ω extensible automorphisms.
- If there is any automorphism property such that an automorphism of a subgroup satisfying the property lifts to an automorphism of the group satisfying the property, then it implies ω extensibility. In particular, the property of being an inner automorphism is stronger than the property of ω extensibility.

- One way of establishing that the only extensible automorphisms are inner is to find a *single* embedding into a group to which the only automorphisms that extend are inner. This was the essence of the formulation of problem 3.
- A related problem is to find a *single* embedding with the property that no automorphism in the bigger group takes an element in the smaller group to an element in a *different conjugacy class* in the smaller group. This was the essence of the formulation of problem 4. It is neither stronger nor weaker than problem 3.
- The final formulation (which is the strongest of all) says that any partial isomorphism of the smaller group that lifts to an automorphism of the bigger group must lift to an inner automorphism of the smaller group. This was the essence of the formulation of problem 5.

2. A possible construction strategy

2.1. **Splinchers.** This construction was based on an idea of Dr. Isaacs for the case of Abelian groups. A **splincher** for a group H is a group G along with an embedding of H in the automorphism group of G, such that:

- H is a **central factor** of the automorphism group of G. [3]
- G is a **characteristic subgroup** [4] in its **semidirect product**[5] with H under the induced action of H on G.

Claim 1. *The embedding of a group in the semidirect product of its splincher with it, satisfies the conditions of Problem 5.*

The proof basically relies on a very simple idea: "automorphisms of a group act on the automorphism group by inner automorphisms on it". We see this idea routinely when looking at linear transformations. A linear transformation on a vector space acts on the space of linear transformations via an inner automorphism by that same element. [6]

Proof. Let H be the group and G be a splincher. Further, suppose that K is the semidirect product. Then, we need to show that if an automorphism of G induces an isomorphism of two subgroups of H, then that isomorphism can also be induced by an inner automorphism of H.

Let σ be an automorphism of K. Because G is a characteristic subgroup of K, σ induces an automorphism on the quotient group of K by G, which is abstractly isomorphic to H.

Applying σ to an element of H takes it to another element of H, times some element in G. The element of H to which it goes corresponds to the image in the action on H as a quotient group. Let $\sigma'(h)$ denote the element on H and $\rho(h)$ denote the element in G. Thus, we have:

$$\sigma(h) = \sigma'(h)\rho(h) \text{ where } h \in H$$

Now, consider any $g \in G$ and $h \in H$. Clearly, as σ is an automorphism of K:

[3] A central factor is a subgroup such that every inner automorphism of the bigger group restricts to an inner automorphism of the subgroup. The definition typically used – that of being a factor in a central product, is equivalent.

[4] A subgroup is characteristic if every automorphism of the group leaves the subgroup invariant.

[5] A semidirect product is obtained by one group acting on another, by considering a group where these two groups intersect trivially and the action is effected through inner automorphisms.

[6] That is, if we have ψ as a linear transformation, then the action of ψ on another linear transformation ϕ must be such that $\psi(\phi)$ acting on $\psi(x)$ is the same as ψ acting on $\phi(x)$. Rearranging this expression gives that $\psi.\phi$ is conjugation of ϕ by ψ.

$$\sigma(h)\sigma(g)\sigma(h)^{-1} = \sigma(hgh^{-1})$$

Writing $\sigma(h)$ as $\sigma'(h)\rho(h)$ we get:

$$\sigma'(h)(\rho(h)\sigma(g)\rho(h)^{-1})\sigma'(h)^{-1} = \sigma(hgh^{-1})$$

Now, h and $\sigma'(h)$ are elements in H, so that conjugating by them is simply their action on G. This enables us to rewrite the above as:

$$\sigma'(h).(\rho(h)\sigma(g)\rho(h)^{-1}) = \sigma(h.g)$$

As this is true for all g, we can rewrite this as (here h denotes h as an automorphism on G:

$$(\sigma'(h)).(\text{Conjugation by } \rho(h)).\sigma = \sigma.(h)$$

Multiplying both elements by σ^{-1} we get:

$$\sigma'(h) = \sigma.h.\sigma^{-1}.(\text{Conjugation by } \rho(h))^{-1}$$

The upshot is that σ' acts on H by conjugating it by some element in the automorphism group of G, modulo some elements of the inner automorphism group of G.

In particular if an element h in the section H goes to another element in the section H, then we get:

$$\sigma'(h) = \sigma.h.\sigma^{-1}$$

Thus, if there is an isomorphism between two subgroups of H that arises via σ, then it is given by an inner automorphism in the automorphism group of G (note that σ,though initially treated as an automorphism of K, is currently being viewed as an automorphism of G).

Now, because H is a central factor of the automorphism group of G, this inner automorphism of the automorphism group of G can be restricted to an inner automorphism of H, completing the proof.

□

Call a group **splinchable** if it possesses a splincher. Then splinchable groups satisfy the condition of problem 5.

2.2. **Splinchers for direct products.** Any group that possesses a splincher satisfies the conditions of problem 5 and hence every automorphism of such a group is inner. Thus, if we somehow are able to construct a splincher for every group, then we have solved all the problems.

We begin by proving an important claim on splinchers for direct products:

Claim 2. *If a finite group can be written as a direct product of groups each of which has a splincher, such that all the splinchers have order relatively prime to each other and to the group, then the group has a splincher, namely the direct product of the splinchers with component wise action.*

Proof. Let:

$$H = H_1 \times H_2 \dots H_n$$

be the direct product decomposition. Let G_i be a splincher for H_i so that all the G_i have relatively prime order to each other and to H. Then, consider the group $G_1 \times G_2 \dots G_n$ with the action of H on it given component wise.

We need to show that this is a splincher. In particular we need to check two conditions:

- The above gives an embedding of H in the automorphism group of G where H is a central factor of the automorphism group.
- G is characteristic in the semidirect product of G by H.

By the order considerations, it is clear that G, and moreover, each G_i, is characteristic. It is also clear that the map from H to the automorphism group of G is an embedding.

It is only required to show that H is a central factor of the automorphism group of G. First, we observe that because the G_i are relatively prime to each other, any automorphism of G must induce an automorphism on each G_i. Conversely, given an automorphism of each G_i, we can retrieve an automorphism of G. We can see that this gives a bijection from the direct product of the automorphism groups of the G_i and the automorphism group of G.

Thus, any inner automorphism by an element in the automorphism group of G boils down to component wise inner automorphisms in the automorphism groups of each G_i. Use the fact that H_i is a central factor in each, we can get an element in H that induces the same inner automorphism in H as the element in the automorphism group of G does. □

2.3. **Splinchers for cyclic and Abelian groups.**

Claim 3. *If $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group of order n, then the additive group of any prime field F_p (that is, $\mathbb{Z}/p\mathbb{Z}$) with $n|p-1$ is a splincher, with the natural action of $\mathbb{Z}/n\mathbb{Z}$ via multiplication by n.*

Proof. The condition of characteristicity follows from the fact that the orders are relatively prime. The condition that $\mathbb{Z}/n\mathbb{Z}$ is a central factor follows trivially from the fact that the automorphism group is simply $\mathbb{Z}/(p-1)\mathbb{Z}$ which is Abelian. □

Using the fact that there are infinitely many primes congruent to 1 modulo any n, and the fact that two distinct primes are relatively prime, we get the following easy corollary:

Claim 4. *Every Abelian group has an Abelian splincher.*

Proof. By the Structure Theorem for Abelian groups, every Abelian group is a direct product of cyclic groups. By the above remark we can determine splinchers for each of the cyclic groups with relatively prime orders to each other and to the original group. Then, by the result on direct products of splinchers, their direct product becomes a splincher for the original group. This splincher is Abelian because it is a direct product of cyclic groups. □

Unwinding our chain of reasoning, we have the following fact: "in an Abelian group, we can embed it in a group such that no automorphism of the bigger group takes one subgroup of the Abelian group to another subgroup of the Abelian group." In particular "there is no automorphism of the bigger group taking an element of the Abelian group to another". Also, "there is no nontrivial automorphism of the Abelian group that extends to an automorphism of the bigger group".

2.4. **Groups with Abelian splinchers.** What are the groups that possess Abelian splinchers? Reversing the question, construct all central factors of automorphism groups of Abelian groups, where the Abelian group is characteristic in the semidirect product. Clearly, in order to study this problem it is necessary to first understand the structure of automorphism groups of Abelian groups. More specifically, we can begin by understanding the structure of the automorphism groups of Abelian groups of prime power order.

Let us begin with the elementary Abelian groups. The automorphism group of an elementary Abelian group of order p^k is the general linear group of order k over F_p, typically denoted as $GL_n(F_p)$. We prove that:

Claim 5. *Every elementary Abelian group is characteristic in its semidirect product with its automorphism group.*

Proof. Clearly the elementary Abelian group is a minimal normal subgroup of the semidirect product (because there is a linear transformation mapping any subspace of it to another subspace with the same dimension). Thus, if there is any automorphism that does not take it to itself, it must take it to some other minimal normal subgroup intersecting it trivially. But then the two normal subgroups would commute element wise, and we know that no element outside the elementary Abelian group commutes with every element of it.

So, every automorphism must take the elementary Abelian group to itself. □

Thus $GL_n(F_p)$ possesses an Abelian splincher.

2.5. **Direct products of Abelian and linear groups.** Combining the ideas obtained so far, we can conclude that:

Claim 6. *A direct product of general linear groups over different primes, and an Abelian group whose order is relatively prime to all the general linear groups and their underlying primes, has an Abelian splincher.*

2.6. **Complete groups.** There are some groups for which splinchers are not needed to establish that they satisfy the conditions of problem 5. These are the groups where every automorphism is inner. Clearly, for such groups, the problem conditions are trivially satisfied.

Centerless groups where every automorphism is inner are termed **complete groups**. In particular, the symmetric groups S_n where $n \neq 2, 6$ are complete.

3. Further questions

3.1. **On splinchers.** We have seen the following properties, each implying the one below it:

(1) The property of having an Abelian splincher
(2) The property of having a splincher, or of being splinchable
(3) The property of satisfying the conditions of problem 5
(4) The property of satisfying the conditions of problem 3
(5) The property that every extensible automorphism of the group is inner
(6) The property that every ω extensible automorphism of the group is inner

Conjectures can be formulated at various levels. We may conjecture that all groups have Abelian splinchers, which is the strongest possible statement. We may conjecture that all groups are splinchable, which is somewhat weaker. We may conjecture that all groups satisfy the condition of problem 5, which is still weaker.

Possible areas of further work are:

- Can we show any of the above properties to be equivalent?
- Can we determine the classes of groups that satisfy the above properties? For instance, we have shown that all Abelian groups and general linear groups have Abelian splinchers.
- Can we formulate alternative properties that may lead to a resolution of problem 1?

3.2. **Consequences of extensible automorphisms.** Another line of study is to look at the property of automorphisms being extensible and k extensible and try to study new subgroup properties defined in terms of these.

In fact, a large number of proofs of elementary facts involving inner automorphisms only make use of the fact that inner automorphisms can be canonically extended in a commuting manner. For instance, consider the following statement:

> Let G be a group. If $H_1 \trianglelefteq K_1 \leq G$ and $H_2 \trianglelefteq K_2 \leq G$ then $H_1 \cap H_2 \trianglelefteq K_1 \cap K_2$. That is, if we have two pairs of a subgroup and its **normal subgroup** then the intersection again gives a pair of a group and its normal subgroup.

The link between normality and inner automorphisms is that normality is the property of invariance under inner automorphisms. The proof of the statement then uses the fact that any inner automorphism of $K_1 \cap K_2$ can be lifted to an automorphism of K_1 on the one hand and K_2 on the other.

Thus, if we replace normality by the property of being invariant under all ω extensible automorphisms, the proof goes through.

Perhaps, by studying the consequences in greater detail, we may stumble across a proof, or a counterexample, to the statement that all extensible automorphisms are inner.

3.3. **A logical formulation.** This looks at the philosophical question : "why are inner automorphisms extensible?" Clearly, because the map $g \mapsto hgh^{-1}$ is an automorphism by design. Thus, those *formulas* that give rise to automorphisms in any structure will automatically lift.

Let us restrict to a variety of universal algebras. Define a formula (expression with some parameters and some variables, using the universal algebra operations) to be **conditionally automorphing** (*my own terminology*) if it is in terms of some parameters, and if the formula is an automorphism whenever some equations are satisfied between the parameters. For instance, in a unital (not necessarily commutative) ring, the map $s \mapsto xsy$ is a conditionally automorphing formula under the condition that $xy = yx = 1$.

It is easy to see that any automorphism that arises via a conditionally automorphing formula extends to an automorphism satisfying the same formula for any embedding, because the universal equations continue to hold. Thus, these are canonically ω extensible.

In the case of groups, the only conditionally automorphing formulas are, in essence, the inner automorphism formulas. Thus, the question that we are asking for groups is an instance of the more general question: "is every extensible automorphism one arising from a conditionally automorphing formula?"

The answer is clearly negative when the structure is very weak. For instance, in the variety with no operations, that is, the variety of sets, any permutation is an ω extensible automorphism, but clearly the only *formulas* possible are for constant maps or the identity map (because there are no operations).

The question thus is: "how strong is the structure of groups? Is it strong enough that all extensible automorphisms are captured by conditionally automorphing formulas?"

3.4. **Analogous question for the variety of Abelian groups.** The question for the variety of Abelian groups is as follows: "what are the automorphisms of an Abelian group that lift to automorphisms of all Abelian groups containing it?" Note that the problem we have solved for the case of groups is the automorphisms that lift to automorphisms of *all* groups containing it, while the problem we now consider is of automorphisms that extend to automorphisms of all *Abelian* groups containing it.

In this case, the *multiplication maps* are conditionally endomorphing, though not conditionally automorphing. This question is interesting and possibly easier to solve. Some progress has been made on it that I am not discussing here.

3.5. **General homomorphisms.** We have so far been interested in determining those automorphisms that gave rise to automorphisms for embeddings into other groups. Embeddings are just injective homomorphisms. Thus, we have looked at automorphisms that give rise to automorphisms over injective homomorphisms. A more general question might be to find automorphisms that gave rise to automorphisms for all homomorphisms. That is:

Problem 6. *Find all automorphisms σ of a group G, such that for any homomorphism ρ from G to a group H, there is an automorphism σ' on H such that $\sigma'.\rho = \rho.\sigma$.*

If we require ρ to be injective, then we get precisely the definition of extensible automorphism. Thus the above property is *a priori* somewhat stronger than that of being extensible. We shall call automorphisms that satisfy the conditions of the above problem **homomorphism transferable**.(*my own terminology*)

It is easy to see that an automorphism is homomorphism transferable if and only if it is extensible and it satisfies the above condition for surjective homomorphisms. Automorphisms that satisfy the condition for surjective homomorphism shall be called **quotientable automorphisms**.(*my own terminology*)

Clearly, inner automorphisms, or automorphisms arising via conditionally automorphing formulas, are homomorphism transferable.

What can we say about quotientable automorphisms? Indeed, there are quotientable automorphisms that are not inner. This follows from the observation below.

Claim 7. *Any automorphism that can be described by some "algebraic formula" in terms of the universal algebra operations using one or more parameters, is quotientable. In fact, the automorphism on the quotient is given by the same algebraic formula with the parameters being the images of the original parameters via the quotient map.*

Call an automorphism **algebraically formulable**(*my own terminology*)if it can be expressed via an algebraic formula. Note that the set of algebraically formulable automorphisms is closed under composition though it is not clear whether it is closed under inversion.

This gives rise to the final problem that needs to be explored:

Problem 7. *Given a group, what can be said about the collection of quotientable automorphisms? Is every quotientable automorphism algebraically formulable?*

4. Inspiration and acknowledgement

4.1. **Inspiration for the problem.** The problem came to me through a formalism I have developed called **property theory**, whereby group properties and subgroup properties are subjected to manipulation and studied under these manipulations. For instance, expressing subgroup properties using the $\leftarrow$ notation (introduced in section 1.3) helps us to uncover some facts about them.

4.2. **The partial solution so far.** The solution uncovered so far has largely been due to the kind help provided by Dr. I. M. Isaacs, who furnished the first example of the non-Abelian group of order 21, and also provided the example of $GL_3(F_2)$. He also suggested some approaches to generalizing the problem and his suggestions led to my formulating problems 4 and 5. He gave the proof for the case of cyclic groups and encouraged me to settle the case for Abelian groups before trying to proceed further.

The conversion of the specific proofs (for the Abelian case) into general ideas of splinchers and central factors was done by me to determine how the approach can be generalized.

4.3. **Alternative approaches.** Prior to discussing the problem with Dr. Isaacs, I also discussed it with Professor Ramanan, who tried an alternative approach using character theory. However, the approach met with some roadblocks. The rough idea was to use the fact that two elements not in the same conjugacy class differ for some class function and hence we can obtain a character where they take different values. Starting with this observation, we try to construct a group, by a slight variation of the general linear group, where there is no automorphism taking one element to the other.

I have worked on this approach as well without much success. However, it has led me to reformulate the approach in a manner that can be reconciled with the splincher approach. This will be discussed later separately.

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