Trigonometric functions and Fourier series

Vipul Naik

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Outline

Periodic functions on reals

Homomorphisms

Fourier series in complex numbers language
  Quick recap

Rollback to real language

What do we mean by infinite sum?
  A little vector space theory
  Infinite sums in a vector space

The inner product space of periodic functions
  Definition of an inner product

Finding a basis
  A basis for the inner product

Finding the coefficients
  General theory in inner product spaces
  In the case of Fourier series
Periodic functions

Let $X$ be a set and $f : \mathbb{R} \to X$ be a function. Then a number $h \in \mathbb{R}$ is termed a **period** for $f$ if for any $x \in \mathbb{R}$:

$$f(x + h) = f(x)$$
Periodic functions

Let $X$ be a set and $f : \mathbb{R} \to X$ be a function. Then a number $h \in \mathbb{R}$ is termed a period (defined) for $f$ if for any $x \in \mathbb{R}$:

$$f(x + h) = f(x)$$

A fundamental period (defined) for $f$ is a positive period such that there exists no smaller positive period.
The period group

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This is termed the **period group** (defined) of \( f \).
What kind of subgroups?

Suppose $f$ is a continuous function, and further suppose that $h_1, h_2, \ldots$, is a sequence of periods of $f$. Then, for any $x$:

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This gives:

$$f(x) = f(x + h)$$
Period groups of continuous functions are closed

The upshot of the previous slide is that the limit of any sequence of periods is also period.
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The upshot of the previous slide is that the limit of any sequence of periods is also period. Hence, the period group of the continuous function on $\mathbb{R}$ is a closed subgroup of $\mathbb{R}$. 
What are the closed subgroups of \( \mathbb{R} \)?

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- The whole of \( \mathbb{R} \)
- A subgroup of the form \( m\mathbb{Z} \) where \( m \in \mathbb{R} \), viz a discrete subgroup comprising integral multiples of \( m \neq 0 \)
- The trivial subgroup, that is, the subgroup comprising only the zero element
Classification of continuous functions based on period group

A continuous function, based on its period group, can be classified as:

- A **constant function** (defined): The period group is $\mathbb{R}$
- A **periodic function** (defined): The period group is $m\mathbb{Z}$ for $m > 0$. This $m$ is the fundamental period
- A non-periodic function: The period group is trivial
In the language of groups

Any function on $\mathbb{R}$ can also be viewed as a function on the coset space of its period group. That’s because by its very definition, the period group is the group such that the function is constant on every coset.
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Any function on $\mathbb{R}$ can also be viewed as a function on the coset space of its period group. That’s because by its very definition, the period group is the group such that the function is constant on every coset. Since $\mathbb{R}$ is an Abelian group, the coset space is actually a group. Thus, the study of periodic continuous functions on $\mathbb{R}$ is equivalent to the study of continuous functions on the group $\mathbb{R}/m\mathbb{Z}$.
The coset space is the circle group

Some observations:

- By composing on the right with an appropriate scalar multiplication, we can *normalize* the period to some value, say $2\pi$. That is, if $f$ has period $m$, the map $x \mapsto f(2\pi x/m)$ has period $2\pi$. 
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- By composing on the right with an appropriate scalar multiplication, we can *normalize* the period to some value, say $2\pi$. That is, if $f$ has period $m$, the map $x \mapsto f(2\pi x/m)$ has period $2\pi$.

- Suppose $m = 2\pi$. Then consider the map $\mathbb{R}$ to $S^1$ that sends $x \in \mathbb{R}$ to $(\cos x, \sin x)$. Clearly, this map is periodic with fundamental period $2\pi$. Further, if $S^1$ is viewed as a group of rotations, the map is an isomorphism.
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- Thus, the study of periodic continuous functions on $\mathbb{R}$ is the same as the study of continuous functions on $S^1$. 
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Homomorphisms from $\mathbb{R}$ to $\mathbb{R}$

Question: What are the *continuous* homomorphisms from $\mathbb{R}$ to itself (as a group)?
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Since $\mathbb{Q}$ is dense in $\mathbb{R}$, any continuous homomorphism from $\mathbb{R}$ to itself is completely determined by its behaviour on $\mathbb{Q}$. Thus, it suffices to determine the possible homomorphisms from $\mathbb{Q}$ to $\mathbb{R}$.
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By group-theoretic considerations, any homomorphism from $\mathbb{Q}$ to $\mathbb{R}$ is of the form $x \mapsto \lambda x$. Hence, any homomorphism from $\mathbb{R}$ to $\mathbb{R}$ is also of the form $x \mapsto \lambda x$. 
Homomorphisms from $\mathbb{R}$ to $S^1$

The picture is like this:

\[
\begin{array}{cc}
\mathbb{R} & \mathbb{R} \\
\downarrow & \downarrow \\
S^1 & S^1
\end{array}
\]

The downward maps are the quotient maps.
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The question: given a continuous homomorphism from $\mathbb{R}$ to $S^1$ (top left to bottom right) can we obtain a continuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$ (top left to top right) such that the diagram commutes?

Answer: Yes

Thus any homomorphism from $\mathbb{R}$ to $S^1$ looks like $x \mapsto (\cos \lambda x, \sin \lambda x)$.  

### Proof

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous homomorphism.  Then we have a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{f} & \mathbb{R} \\
\downarrow & & \downarrow \\
S^1 & \xrightarrow{\phi} & S^1
\end{array}
\]

where $\phi$ is the quotient map.  Now consider the map $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(\phi^{-1}(x))$.  This map is continuous and commutes with the quotient map $\phi$, hence it is a homomorphism.  Thus any continuous homomorphism from $\mathbb{R}$ to $S^1$ looks like $x \mapsto (\cos \lambda x, \sin \lambda x)$.  

### Example

Consider the homomorphism $f: \mathbb{R} \to S^1$ defined by $f(x) = e^{2\pi i x}$.  Then $f$ is a continuous homomorphism from $\mathbb{R}$ to $S^1$.  We can construct a homomorphism $g: \mathbb{R} \to \mathbb{R}$ by taking $g(x) = f(\phi^{-1}(x))$.  This map is continuous and commutes with the quotient map $\phi$, hence it is a homomorphism.  Thus any continuous homomorphism from $\mathbb{R}$ to $S^1$ looks like $x \mapsto (\cos \lambda x, \sin \lambda x)$.  

### Conclusion

We have shown that any continuous homomorphism from $\mathbb{R}$ to $S^1$ looks like $x \mapsto (\cos \lambda x, \sin \lambda x)$.  This is a useful result in the study of harmonic analysis and has applications in various fields of mathematics, including number theory, ergodic theory, and quantum mechanics.  

### Further Reading

Homomorphisms from $\mathbb{R}$ to $S^1$

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Homomorphisms from \( S^1 \) to \( S^1 \)

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The downward maps are the quotient maps.
Given a continuous homomorphism $\phi : S^1 \to S^1$, composing $\phi$ on the right with the projection from $\mathbb{R}$ to $S^1$ gives a continuous homomorphism from $\mathbb{R}$ to $S^1$. 
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This combined with the previous result, tells us that any homomorphism from $S^1$ to $S^1$ is of the form:

\[(\cos x, \sin x) \mapsto (\cos nx, \sin nx)\]
In terms of complex numbers

The plane $\mathbb{R}^2$ can be viewed as the complex numbers $\mathbb{C}$, that is, every point $(x, y)$ can be identified with the complex number $x + iy$. Under this identification, $S^1$ is a subgroup of the multiplicative group of nonzero complex numbers.
In terms of complex numbers

The plane $\mathbb{R}^2$ can be viewed as the complex numbers $\mathbb{C}$, that is, every point $(x, y)$ can be identified with the complex number $x + iy$. Under this identification, $S^1$ is a subgroup of the multiplicative group of nonzero complex numbers. In this language, then $(\cos x, \sin x)$ is the same as $e^x$. Thus, homomorphisms from $S^1$ to $S^1$ are maps of the form:

$$e^x \mapsto e^{nx}$$

which is the same as:

$$z \mapsto z^n$$
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1. We wanted to understand continuous periodic functions from $\mathbb{R}$ to $X$. 
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2. We converted this to understanding continuous functions from \( \mathbb{R}/2\pi\mathbb{Z} \) to \( X \).
3. We converted this to understanding continuous functions from \( S^1 \) to \( X \).
What we’re interested in

The situation that we want to study is where $X = \mathbb{C}$, viz the problem of continuous periodic functions from $\mathbb{R}$ to $\mathbb{C}$. This reduces to the problem of all continuous functions from $S^1$ to $\mathbb{C}$.
What we’re interested in

The situation that we want to study is where $X = \mathbb{C}$, viz the problem of continuous periodic functions from $\mathbb{R}$ to $\mathbb{C}$. This reduces to the problem of all continuous functions from $S^1$ to $\mathbb{C}$.

We already have a bunch of continuous functions from $S^1$ to $S^1$ (which is a subset of $\mathbb{C}$), namely: the maps $z \mapsto z^n$. These maps are called characters.
Are all continuous functions expressible via characters?

Which continuous maps from $S^1$ to $\mathbb{C}$ can be expressed in terms of characters?
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- Polynomial maps are expressible as finite linear combinations of characters.
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Which continuous maps from $S^1$ to $\mathbb{C}$ can be expressed in terms of characters?

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- Laurent polynomial maps are expressible as \textit{finite} linear combinations of characters
Are all continuous functions expressible via characters?

Which continuous maps from $S^1$ to $\mathbb{C}$ can be expressed in terms of characters?

- Polynomial maps are expressible as \textit{finite} linear combinations of characters
- Laurent polynomial maps are expressible as \textit{finite} linear combinations of characters
- Maps which have expressions as power series or as Laurent series about the origin, can be expressed as \textit{infinite} linear combinations of characters

Do all continuous maps fall in the third class? Can every continuous map on $S^1$ be expressed as a (possibly infinite) sum of characters?
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Continuous functions to reals

Let’s now consider the case where $X = \mathbb{R}$, that is, the problem of determining continuous functions from $S^1$ to $\mathbb{R}$. Clearly, we cannot *directly* use characters since characters are maps from $S^1$ to $\mathbb{C}$. However, we can *project* the characters on the axes, viz take their coordinates, and obtain the following collections of periodic functions:
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$$z \mapsto \text{Re } z^n \text{ and } z \mapsto \text{Im } z^n$$

for $n \in \mathbb{Z}$
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Continuous functions to reals

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\[ z \mapsto \text{Re} \, z^n \quad \text{and} \quad z \mapsto \text{Im} \, z^n \]

for \( n \in \mathbb{Z} \)

By thinking of these as periodic functions on \( \mathbb{R} \), we get the bunches:

\[ x \mapsto \cos nx \quad \text{and} \quad x \mapsto \sin nx \]
The question for reals is: given a periodic function \( \mathbb{R} \to \mathbb{R} \), what are the conditions under which it can be expressed as an infinite linear combination of the cosine and sine bunches?
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What is a vector space?

Let $k$ be a field (such as $\mathbb{R}$, $\mathbb{C}$). Then a vector space over $k$ (also, a $k$-vector space) is a set $V$ equipped with:

- An additive operation $+$ under which $V$ is an Abelian group
**What is a vector space?**

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- A *scalar multiplication* action of $k$ on $V$ with the property that the scalar multiplication by any element induces an Abelian group homomorphism on $V$
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- An additive operation $+$ under which $V$ is an Abelian group
- A scalar multiplication action of $k$ on $V$ with the property that the scalar multiplication by any element induces an Abelian group homomorphism on $V$

We can thus talk of finite $k$-linear combinations on $V$. There is, however, no inherent meaning associated to infinite $k$-linear combinations on $V$. 
Linear independence and basis

A subset $S$ of a vector space $V$ is said to be *linearly independent* if 0 cannot be expressed as a nontrivial $k$-linear combination of any finite subset of $S$, that is, if:

$$\sum_{i=1}^{r} a_i v_i = 0$$

for $v_i \in S$ and $a_i \in k$, then each $a_i = 0$

The *span* (defined) of a subset $S$ is defined as the vector subspace of $V$ containing all those elements that are finite $k$-linear combinations of elements of $S$.

A *basis* (defined) of a vector space is a linearly independent subset whose span is the whole vector space.
An **inner product** (defined) on a real vector space is a generalization of the dot product that we’ve usually seen. (definition later).
Inner products and orthogonal vectors

An **inner product**\(^{(\text{defined})}\) on a real vector space is a generalization of the dot product that we’ve usually seen. (definition later).

Two vectors are said to be **orthogonal**\(^{(\text{defined})}\) if their inner product is zero.
Inner products and orthogonal vectors

An **inner product** (defined) on a real vector space is a generalization of the dot product that we’ve usually seen. (definition later).

Two vectors are said to be **orthogonal** (defined) if their inner product is zero.

Now for an observation: any family of elements that is pairwise orthogonal with respect to an inner product, is also linearly independent.
Orthonormal basis

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An orthonormal basis is a basis where the elements are pairwise orthogonal.

Given a vector space and an inner product, can we find an orthonormal basis for the vector space?
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Given a vector space and an inner product, can we find an orthonormal basis for the vector space?
The answer is always yes for a finite-dimensional vector space. The standard technique is Gram-Schmidt orthogonalization.
Topological vector space

A topological vector space is a vector space with a topology given both to the vector space and to the base field such that:

- All the field operations are continuous with respect to the topology given to the base field.
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A topological vector space is a vector space with a topology given both to the vector space and to the base field such that:

- All the field operations are continuous with respect to the topology given to the base field.
- The group operations on the vector space are continuous with respect to the topology on the vector space.
Topological vector space

A topological vector space is a vector space with a topology given both to the vector space and to the base field such that:

- All the field operations are continuous with respect to the topology given to the base field.
- The group operations on the vector space are continuous with respect to the topology on the vector space.
- The scalar multiplication operation is continuous from the product of the field and the vector space to the vector space.
Infinite sums in a topological vector space

Given a sequence of vectors $v_1, v_2, \text{ and so on}$ in a topological vector space, the infinite sum $\sum_{i=1}^{\infty} v_i$ is defined as the \textit{limit} of the partial sums:
Infinite sums in a topological vector space

Given a sequence of vectors $v_1$, $v_2$, and so on in a topological vector space, the infinite sum $\sum_{i=1}^{\infty} v_i$ is defined as the \textit{limit} of the partial sums:

$$\sum_{i=1}^{\infty} v_i = \lim_{n \to \infty} \sum_{i=1}^{n} v_i$$
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\sum_{i=1}^{\infty} v_i = \lim_{n \to \infty} \sum_{i=1}^{n} v_i
\]

Note that the infinite sum makes *no sense* without the topology because it depends on the notion of limit.
Normed linear space

Consider an \( \mathbb{R} \)-vector space \( V \). A norm function on this vector space associates to each \( v \in V \) a nonnegative real number \( N(v) \) such that:

- \( N(v) = 0 \implies v = 0 \)
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- \( N(\lambda v) = |\lambda| N(v) \)
Normed linear space

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- $N(v) = 0 \implies v = 0$
- $N(\lambda v) = |\lambda| N(v)$
- The map $(v, w) \mapsto N(v - w)$ defines a metric on $V$. Equivalently, for any vectors $v, w \in V$:

\[
N(v + w) \leq N(v) + N(w)
\]
The $L^r$-normed linear spaces

Examples of norms are the $L^r$-norms, in finite-dimensional vector spaces. Take a basis $e_1, e_2, \ldots, e_n$ of $V$ and for any $v \in V$, consider the unique expression:

$$v = \sum_{i=1}^{n} \lambda_i e_i$$

Now define:

$$N(v) := \left( \sum_{i=1}^{n} \lambda_i^r \right)^{1/r}$$
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For $r \geq 2$, $N$ is a norm.
Any normed $\mathbb{R}$-vector space automatically gets the structure of a topological vector space, because, first of all, it gets the structure of a *metric space*, and any metric space naturally comes with a topology.
Any normed $\mathbb{R}$-vector space automatically gets the structure of a topological vector space, because, first of all, it gets the structure of a metric space, and any metric space naturally comes with a topology. Thus, we can talk of notions of infinite sums and convergence for normed vector spaces.
Inner product space

Let $V$ be a $\mathbb{R}$-vector space. Then an inner product on $V$ associates to every pair of vectors $v, w \in V$ a real number $\langle v, w \rangle$ such that:

- It is linear in the first variable:
  
  \[
  \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle
  \]
  
  and
  
  \[
  \langle \lambda v, w \rangle = \lambda \langle v, w \rangle
  \]
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  \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle
  \]
  and
  \[
  \langle \lambda v, w \rangle = \lambda \langle v, w \rangle
  \]

- It is symmetric, viz:
  \[
  \langle v, w \rangle = \langle w, v \rangle
  \]
Inner product space

Let \( V \) be a \( \mathbb{R} \)-vector space. Then an inner product on \( V \) associates to every pair of vectors \( v, w \in V \) a real number \( \langle v, w \rangle \) such that:

▶ It is linear in the first variable:

\[
\langle v_1, w \rangle + \langle v_2, w \rangle = \langle v_1 + v_2, w \rangle
\]

and

\[
\langle \lambda v, w \rangle = \lambda \langle v, w \rangle
\]

▶ It is symmetric, viz:

\[
\langle v, w \rangle = \langle w, v \rangle
\]

▶ It is positive definite, viz:

\[
\langle v, v \rangle > 0
\]

for \( v \neq 0 \)
Inner product spaces are normed

Given a space $V$ with an inner product, we can naturally define a norm on $V$ by setting $N(v) = \sqrt{\langle v, v \rangle}$. 
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Given a space $V$ with an inner product, we can naturally define a norm on $V$ by setting $N(v) = \sqrt{\langle v, v \rangle}$. Thus, every inner product space is a normed space and hence also a topological vector space.
Concerns when dealing with infinite sums

Having seen a bit of infinite sums on the real line, we should be aware of the different notions of convergence:

1. **absolute convergence**\(^{(\text{defined})}\): Here the series of norms of the vectors is convergent.
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Having seen a bit of infinite sums on the real line, we should be aware of the different notions of convergence:

1. **absolute convergence**\textsuperscript{(defined)}: Here the series of norms of the vectors is convergent.

2. **unconditional convergence**\textsuperscript{(defined)}: Here any rearrangement of the vectors gives a convergent series.

3. **conditional convergence**\textsuperscript{(defined)}: Here, the given series is convergent, but nothing is guaranteed about rearrangements of it.
A Cauchy sequence in a topological vector space is a sequence where for any bound $\epsilon$, there exists an $N$ such that for all $m, n > N$, the tail sum $S_m - S_n$ is bounded in magnitude by $\epsilon$. Any convergent sequence is Cauchy. If, in a normed vector space, the converse is true (viz any Cauchy sequence is convergent) then the normed vector space is said to be complete (defined). For instance, any finite-dimensional vector space over $\mathbb{R}$ is complete.
The use of Cauchy sequences

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Trigonometric functions and Fourier series

Vipul Naik

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  In the case of Fourier series
Inner product defined by an integral

Consider the following integral for functions $f, g : S^1 \to \mathbb{R}$:

$$\int_{S^1} f(x) g(x) \, dx$$

where the measure of integration is the usual arc-length function on $S^1$. 
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where the measure of integration is the usual arc-length function on $S^1$. Alternatively, via the identification of $S^1$ with the interval from 0 to $2\pi$, we have:

$$\int_{0}^{2\pi} f(x)g(x) \, dx$$
Some observations about this integral

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Thus, this integral defines an inner product if we restrict to the subspace defined by continuous functions on $S^1$. 
Notion of convergence for functions

Before proceeding to study convergence questions, we must define a suitable notion of convergence for functions. There are the following notions possible:

- Pointwise convergence: This means that the value of the function in the sequence must, at each point, converge to the value of the function.
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Before proceeding to study convergence questions, we must define a suitable notion of convergence for functions. There are the following notions possible:

- **Pointwise convergence:** This means that the value of the function in the sequence must, at each point, converge to the value of the function.

- **Uniform convergence:** This means that for any $\epsilon$, there is a uniform $\delta$ we can choose such that the $\epsilon$-$\delta$ condition holds at every point.

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- **$L^r$-norm convergence:** This convergence arises from the naturally defined $L^r$-norm. The $L^r$-norm in this case is an infinite-dimensional analogue of the $L^r$-norm for finite-dimensional spaces:

$$f \mapsto \left( \int_0^{2\pi} f(x)^r \, dx \right)^{1/r}$$
Uniform convergence and the sup-norm

The supremum of a continuous function from $S^1$ to $\mathbb{R}$ is defined as the maximum of absolute values of elements in its image set. The norm that associates to each continuous function its supremum, gives rise to the topology of uniform convergence.
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For obvious reasons, the sup-norm is also often called the $L^\infty$-norm.
We can further show that under uniform convergence, the limit of any sequence of continuous functions is also continuous. Thus, with respect to the topology of uniform convergence, the continuous functions form a closed subspace.
The inner product and the topology induced

The inner product that we considered:

\[ \int_{0}^{2\pi} f(x)g(x) \, dx \]

... gives rise to the norm:

\[ f \mapsto \sqrt{\int_{0}^{2\pi} f(x)^2 \, dx} \]

Which is the \( L^2 \)-topology. Thus, this inner product induces the topology of \( L^2 \)-convergence.
Relation between these topologies

It turns out that for $r < s$, $L^s$-convergence is a stronger condition than $L^r$-convergence. This is because for $L^s$-convergence, the sequence must converge rapidly to its limits even at points where the deviation is much greater.
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Thus, while results on $L^2$-convergence come for free when dealing with this inner product, results on uniform convergence require additional machinery.
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The cosine and sine functions – orthogonal

The following can be checked:

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\int_0^{2\pi} \cos mx \cos nx \, dx = \pi \delta_{mn}
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The cosine and sine functions – orthogonal

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\[ \int_{0}^{2\pi} \sin mx \sin nx \, dx = \pi \delta_{mn} \]

Thus, the functions \( x \mapsto \cos mx \) and \( x \mapsto \sin nx \) are pairwise orthogonal.
A function $f : S^1 \to \mathbb{R}$ that can be expressed as a finite linear combinations of cos and sin functions is termed a trigonometric polynomial. Given the angle sum formulae, a function is a trigonometric polynomial if and only if it can be expressed as a polynomial in the cos and sin functions. Clearly, then, the finite linear combinations are very few.
Meaning of Fourier series

A Fourier series is an expression as an infinite linear combination of the sine and cosine terms.
Meaning of Fourier series

A Fourier series is an expression as an infinite linear combination of the sine and cosine terms. A typical Fourier series:

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Thus, the question of determining which functions can be expressed as infinite linear combinations reduces to the question of determining which functions have Fourier series that converge to them.
Various kinds of convergence

There are two parts to the question: given a function, can we associate a unique Fourier series to it? If we are able to do that, we can ask:

- For which functions does the Fourier series converge in the $L^2$-norm?
Various kinds of convergence

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- For which functions does the Fourier series converge in the $L^2$-norm?
- For which functions does the Fourier series converge pointwise?
Various kinds of convergence

There are two parts to the question: given a function, can we associate a unique Fourier series to it? If we are able to do that, we can ask:

- For which functions does the Fourier series converge in the $L^2$-norm?
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- For which functions does the Fourier series converge uniformly?
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Expressing a vector using an orthonormal basis

Let \( v_1, v_2, \ldots, v_n \) form an orthonormal basis for a vector space \( V \) with respect to an inner product \( \langle \ , \ \rangle \). Then, if

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v = \sum_i \lambda_i v_i
\]
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\[
v = \sum_i \lambda_i v_i
\]

We have the following formula for \( \lambda_i \):

\[
\lambda_i = \langle v , v_i \rangle
\]

In other words, \( \lambda_i \) is the length of the projection of \( v \) on the \( i^{th} \) coordinate.
Suppose we have an infinite orthonormal set \( v_i \) indexed by \( i \in I \) in a topological vector space \( V \). Then, given a vector \( v \in V \), defines \( \lambda_i = \langle v, v_i \rangle \) as \( i \in I \).
Suppose we have an infinite orthonormal set $v_i$ indexed by $i \in I$ in a topological vector space $V$. Then, given a vector $v \in V$, defines $\lambda_i = \langle v, v_i \rangle$ as $i \in I$.

Question: Under what conditions is it true that:

$$v = \sum_{i \in I} \lambda_i v_i$$
Linearly dense orthonormal set

For an infinite-dimensional topological vector space, we may not be able to find an orthonormal basis. However, we may be able to find an orthonormal set with the property that the infinite linear combinations of elements in that set cover the whole space.
Linearly dense orthonormal set

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This happens if and only if the linear subspace comprising \textit{finite} linear combinations is a dense subspace of the whole space.
For an infinite-dimensional topological vector space, we may not be able to find an orthonormal basis. However, we may be able to find an orthonormal set with the property that the *infinite* linear combinations of elements in that set cover the whole space.

This happens if and only if the linear subspace comprising *finite* linear combinations is a dense subspace of the whole space.

The “abstract nonsense” guarantees that the series obtained in this way converges in the topology induced by the inner product.
Fourier coefficients and $L^2$-convergence

The Fourier series for a $2\pi$-periodic function $f$ looks like:

$$a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$
Fourier coefficients and $L^2$-convergence

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Where we have:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) \, dx$$

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For any $L^2$-function, the Fourier coefficients are well-defined and the Fourier series converges in the $L^2$-sense.
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For any $L^2$-function, the Fourier coefficients are well-defined and the Fourier series converges in the $L^2$-sense. In particular, for continuous functions the Fourier series converges in the $L^2$ sense.
Conditions for uniform and pointwise convergence

It turns out that for a $C^1$ function, the Fourier series not only converges to it in the $L^2$ sense, it also converges uniformly.
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It turns out that for a $C^1$ function, the Fourier series not only converges to it in the $L^2$ sense, it also converges uniformly. Further, for a continuous function, it converges pointwise. We shall see proofs of these facts next time.