AKB TUTORIAL I: UNDERSTANDING DEFINITIONS

VIPUL NAIK

Abstract. This is the first of a ten part tutorial series to accompany the `Assimilate Knowledge Better` articles. This series focuses on developing the requisite competencies for using property theory in knowledge assimilation.

1. What this tutorial covers

1.1. **Why understand definitions?** A fashion these days is to complain that there is "too much stress on theory, and too little on the applications, and that is where we fail". Theory is being treated as something inferior, something cheap done by teachers who do not know better and are unenlightened.

Definitions are treated as a component of the rote system and are part of the "regurgitate and vomit" tradition. Many of the people around me scoff at examinations asking questions such as "Define so-and-so with an example" or "What is a so-and-so?"

Testing today is application oriented. It asks questions that depend on how to apply our knowledge on the ground.

But this does not mean we need to condemn the act of learning and understanding definitions. Testing has become application oriented because:

- The understanding of definitions is difficult to test directly. Rote memorization can be used to "beat the system", or at least, so it is believed. Thus, a testing system that relies on definitions is not sufficiently *discriminatory*.
- *Merely* understanding definitions is not sufficient. It is necessary to be able to apply our understanding to the practical situations that may arise.
- Application oriented problems are easier to cast into objective and multiple choice frameworks. This helps in easier testing.

None of these reasons absolve us of the burden of understanding definitions. In fact, it becomes more important to *understand* the definitions. And here, we are going to see that the process of understanding definitions is, naturally, a very delightful and creative process, and helps set the pace for the rest of learning.

Here is why absorbing and assimilating definitions is crucial:

- Definitions are the way in which new concepts, terms, and encapsulations are introduced. Incorrect understanding of definitions can lead to *wrong concept formation* and *incorrect visualization*. If the definitions are clear in the learner's mind, then the learner has a clear *idea of her knowledge state*.
- The ability to quickly locate and parse definitions helps in the creation of a summary, a learning index, bookmarking, doubt marking. Thus, it *expedites all learning strategies*.
- With the definitions well ingrained, a learner is more effectively able to *communicate her understanding* in a simulated or real testing environment or to other learners.

1.2. **In my experience.** In my experience, people take many harmful and confrontational attitudes towards definitions and this hinders the learning process. There are two attitude problems with definitions:

- The *stutter and stammer* problem: People feel they just will not be able to get the definition right. The definitions slip around and do not seem to stick. The students lament the lack of examples to make the definition *clearer*. This lowers their confidence to approaching the material, giving them the learner's block.
- The *shrug it off* problem: People feel they have read the definition, they remember it, they are ready to memorize it if that serves it any purpose, so what next? Should we move on to the

exercises? The riders? The *applications*? And if they get stuck on the applications, they blame the system for not teaching in an application oriented manner.

Both of these are serious impediments to learning. Because remembering a definition, whether by heart or loosely, serves little purpose unless it is accompanied by an *understanding* of the definition. But even understanding is not enough. We must *appreciate* and *enjoy* the definition.

I have seen myself that as people become more aware of the importance of the definition, they are able to enjoy the learning and concept grasping work more and are *less dependent on attention hooking devices*. They are also able to *derive more value from every example and exercise.*They feel *more in rapport with the learning process*. The *need for instant gratification and entertainment reduces*.

I think that at the root of the mental blocks is the feeling that definitions are part of the hand-me-down, rigid, never changing regime by which teachers torture students. Here, we shall see that definition appreciation can be quite an art. And with that in mind, it becomes clear that definition creation has got to be even more of an art! Definitions are created dynamically, new terms are coined drawing upon plenty of creative resources. Surely this is to be looked up to!

2. Quick Overview of Theory

2.1. The crux of property theory.

Properties are elements in themselves, and need to be given attention as such.

This central idea resounds throughout the `Assimilate Knowledge Better` series, and is the very first realization on the path to appreciating definitions. The idea: most new terms introduced are either properties, or are something related to properties, in ways that we explore here. Heightened sensitivity to these leads to a quicker and thorougher grasp of definitions.

2.2. Properties as we call them.

A **property**(defined) over a collection is something that every object in that collection either *has* or *does not have*. The *collection* over which the property is evaluated is termed the **context space**(defined) of the property. Every property *must* have a context space.

Here are some examples:

- We talk of "malleable materials". Here, the term "malleable" is an adjective connoting a property over the context space of materials. Given a material, we can ask the question: "is it malleable?"
- We talk of "type A personalities". Here, the term "type A" indicates a property over the context space of personalities. Given a personality, we can ask the question "is this a type A personality?"
- We talk of an animal "being a vertebrate". Here, "vertebrate" is a noun denoting a property over the context space of animals. Given an animal, we can ask the question – "is this a vertebrate?"
- We talk of a battery "being fully charged". Here "fully charged" is a property over the context space of batteries in their current state. Given a battery in a given state, we can ask the question – "is this fully charged?"
- We talk of a liquid "having viscosity". Here "viscosity" is a property over the context space of liquids. Given a liquid, we can ask the question – "does this have viscosity?"

2.3. Forms of a property.

The above examples show that a property could be expressed in many forms. These include:

1. The **adjective form**(defined) Here, the property is denoted by an adjective qualifying a noun that denotes a variable instance of the context space. For instance, "malleable material" has "malleable" as an adjective denoting a property over the context space of materials.
2. The **member noun form**(defined): Here, the property is expressed as a noun that describes the label given to an element in the context space having the property. For instance, "vertebrate" is a noun describing the elements having the property of being vertebrates. Similarly, "animal" is a noun denoting the property of being an animal, over the context space of living organisms.
3. The **property noun form**(defined): Here, the property is expressed as a noun, and objects satisfying it are said to *satisfy* or *have* it. For instance, "viscosity", "malleability", "happiness".

There are other forms, in particular the **locative form**. The locative form is the form of a property wherein the property is expressed as "in this class". For instance, "in the league of Hitler" might be a property for rulers, and "in the class of Tendulkar" might be a property over the context space of cricketers.

- The English language talks of **countable nouns**(defined) – nouns that can be given a number, such as boy, hat. The opposite are **uncountable nouns**(defined) or **mass nouns**(defined) – nouns that have quantity, but not number. For instance, water, money and so on. Are member nouns countable or uncountable? And what about property nouns?

2.4. **Four special things about our properties.** The term *property*, as we use it, differs from the loose English usage in four important ways. Of these, the first is very important, while the remaining three are more negotiable:

- **Defined over a context space**: Many adjectives in English are used loosely. We talk of "happy days", "tense rooms", "good tables", "sneaky pillows". Natural language is replete with poetic devices.

  Property theory does not object to these constructs – but it refrains from calling something a property until it has a definite context space. Generic adjectives such as "happy", "good" and "tense" do not qualify as properties until it is clear what context space they are being evaluated over. Until then, they are termed **dangling properties**. Dangling properties are *not* properties.
- **definite**(defined): The property is true or false for every objects. There are no undecidables or in betweens.

  The definiteness rule is not so important at the moment. There are variations to property theory where we consider **fuzzy properties** which are properties that have varying shades of truth. Thus, definiteness, while an important ingredient of property theory, can be dispensed with in some situations.
- **intrinsic**(defined): The property depends intrinsically on the elements in the context space. It should not vary with factors outside the context space. For instance, the property of a coin "that it will give heads when tossed first" is not intrinsic.

  Many properties that at first sight do not seem intrinsic can be made intrinsic if the context space is redefined. For instance, the property of being "angry" is not intrinsic over the context space of persons, because the same person may be angry at one time and not angry at the other. But it is intrinsic when viewed over the context space of the person along with the time.
- **binary**(defined): The property can take only two states, true and false.

  The condition of being binary is also somewhat negotiable. For instance, we may settle for three states such as true, false, and insufficient information. We may settle for a finite number of different states. However, the word *property* is not ideally suited under these circumstances. The word **trait**, with the various states being termed the **trait values**, is more suitable.

2.5. **Strong and weak.** Suppose we have two properties over the same context space. Then one is said to be *stronger* than the other if every element over the context space satisfying the first property also satisfies the second. In other words, the subcollection of the context space satisfying the first property is contained *inside*, or is *smaller than*, the subcollection satisfying the second property.

In symbols, we put this as follows. Let a and b denote the two properties. Then, we say $a \leq b$ or a is *stronger* than b or $a \implies b$ or b is *weaker* than a if any of these equivalent conditions hold:

- Every element satisfying a satisfies b.
- The subcollection satisfying a is contained inside the subcollection satisfying b.
- The condition of satisfying a is more *stringent*, or *demanding*, or *restrictive*, than the condition of satisfying b.
- Satisfying a is sufficient for satisfying b, while satisfying b is necessary for satisfying a.

Note that $a \leq b$ indicates that a is *stronger* than b. The $\leq$ arises because the subcollection satisfying a is indeed smaller than that satisfying b.

We may not always be able to compare two properties. Neither might imply the other. In this case, the properties are *incomparable.* The collection of properties over a context space is termed the **property space**(defined) over it. The property space over a context space is thus equipped with a *partial order* as described above.

For instance, the property of being a mammal is stronger than the property of being warm blooded, when viewed over the context space of animal species, because:

- Every mammal is warm blooded.
- The subcollection of all animal species that are mammal species is contained inside the subcollection of all animal species that are warm blooded.

- The condition of being a mammal is more stringent, restrictive, or demanding than the property of being warm blooded.
- Being a mammal is sufficient for being warm blooded, being warm blooded is necessary for being a mammal.

Thus, we may say: mammal $\leq$ warm blooded or mammal $\implies$ warm blooded.

2.6. **The three parts to a definition.** A little ironically, we must begin by defining what a definition is. A **definition** of a **term** is a statement that gives the authoritative rule based on which all queries regarding that term are to be resolved. This means that *it is all within the definition.* A good definition means a useful and durable formulation of ideas. And a good understanding of existing definitions portends a smooth passage through all the *new* stuff.

Property theory is crucial to definition understanding in a number of ways. One aspect is that *many of the terms defined are properties.* Having learned a few basics of property theory, we have already gotten to one key insight:

Key Point 1. *When a definition is about a new property, then there must be a clearly determined context space for it. So, figuring out the context space is a necessary first step to understanding the definition of the property.*

In an ordinary definition, we just think of a term being defined and a meaning given to it. But when defining properties (which is what a lot of definitions are about) we have to specify not just the property name and the meaning, but also the context space. The definition of a property comprises three parts:

- Property name
- Context space
- Meaning

In the coming sections, we shall become adept at identifying these three parts to the definitions of properties, cutting across all disciplines. The theoretical issues that we have discussed so far, and some others that will come up as we proceed, shall link quite closely with many practical considerations for parsing definitions of properties.

Note that *not all new terms are properties.* So the model we discuss right now will not apply to every definition. There are other kinds of terms, and the later sections shall discuss methods for handling the other kinds of definitions.

3. Parsing single sentence definitions

3.1. **The three parts: a quick recall.** We begin by looking at definitions for **properties**. Not all definitions are definitions of properties – there are many other kinds of definitions. Nonetheless, many definitions fit into the framework below.

A definition such that the context space, property name and meaning are all wrapped into a single sentence is called a **single sentence definition**(defined).

Natural languages offer a variety of tools to put forth single sentence definitions. Consider the following single sentence definitions:

1. An **animal species** `[context space specifier]` where (the female of the species gives milk to the child) `[meaning]` is said to be a *mammalian species* `[property name (member noun)]`.
2. *Anchors* `[property name (member noun)]` are **external stimuli** `[context space specifier]` that (trigger a specific state) `[meaning]`.
3. An **financial asset** `[context space specifier]` is called *liquid* `[property name (adjective)]` if it (can readily be traded for cash) `[meaning]`.
4. An *elastic collision* `[property name (adjective)]` is a **collision** `[context space specifier]` in which (energy is conserved) `[meaning]`.
5. A *cacophony* `[property name (member noun)]` is a **sound** `[context space specifier]` that (does not please the ears) `[meaning]`.
6. A **person** `[context space specifier]` who (does not believe in the existence of God) `[meaning]` is called an *atheist* `[property name (member noun)]`.
7. A **metal** `[context space specifier]` that (can be drawn into long and thin wires) `[meaning]` is said to possess *ductility* `[property name (property noun)]`.
8. *Sadists* `[property name (member noun)]` are **people** `[context space specifier]` who like to (inflict pain, particularly sexual pain, on others) `[meaning]`.

(9) A **finite simple graph**`[context space specifier]` is called *regular*`[property name (adjective)]` if (every vertex has the same degree) `[meaning]`.
(10) A **person** `[context space specifier]` who (talks and behaves politely) `[meaning]` is labeled *well bred* `[property name (adjective)]`.
(11) Two **people** `[context space specifier]` who (get along well with each other) `[meaning]` are said to be *in rapport* `[property name (locative)]`.
(12) A *superconductor* `[property name (member noun)]` is a **material** `[context space specifier]` that (has zero resistivity) `[meaning]`.
(13) An **verb**`[context space specifier]` is termed *inchoate*`[property name (adjective))]` if it (describes the creation, birth, formation or evolution of something) `[meaning]`.
(14) A **positive integer** `[context space specifier]` that (has exactly two positive integer divisors) `[meaning]` is called a *prime* `[property name (member noun)]`.
(15) A **Boolean formula** `[context space specifier]` that (has a satisfying assignment) `[meaning]` is called *satisfiable* `[property name (adjective)]`.
(16) A **decision problem** `[context space specifier]` that (can be solved in polynomial time by a Turing machine) `[meaning]` is said to be *in class P* `[property name (locative)]`.
(17) A *plant* `[property name (member noun)]` is a **living organism** `[context space specifier]` whose (cells have cell walls). `[meaning]`
(18) *Voluptuaries* `[property name (member noun)]` are **people** `[context space specifier]` who (thrive on sensual pleasures). `[meaning]`
(19) A **language** `[context space specifier]` where (morphemes are attached to the words themselves) `[meaning]` is called *agglutinative* `[property name (member noun)]`.
(20) A **formal language** `[context space specifier]` is said to *have property P* `[property name (property noun)]` if (any word in the language can be pumped) `[meaning]`.

Now it is possible that many of these definitions do not make much sense, because they are drawn from diverse areas. Nonetheless, there is a lot of similarity in the sentence structures. In fact, in most of the cases, the sentence structure itself tells us what the context space, property and meaning are, *without* our having any understanding of the subject matter. Broadly, there are three formats:

- The **context space-meaning-property name** format: A `[context space specifier]` (such that)/where `[meaning]` is termed/(said to be)/(called) `[property name]`
- The **context space-property name-meaning** format: A `[context space specifier]` is called/termed `[property name]` provided that `[meaning]`.
- The **property name-context space-meaning** format: A `[property name]` is a `[context space specifier]` such that `[meaning]`.

In each case, the property could be expressed as an adjective, a property noun, a member noun, or a locative.

Concept Testers

(1) For each of the definitions given above, classify into one of the three formats.
(2) A property may be expressible via member nouns, property nouns, and adjectives. Fir instance, the member noun "superconductor" corresponds to the adjective "superconducting" and the property noun "superconductivity". Write definitions of the property in each of these forms, and using each of the formats available.
(3) Look at the definition for the term "single sentence definition". Does it itself fit into any of the formats?

Points to Ponder

- How does the form of the property correlate with the format of the definition? Which format is best suited to the property name being an adjective? A member noun? A property noun?
- Are these formats and forms unique to the English language? Try studying the same phenomena in another natural language and see what things change and what things remain the same.
- One of the special features of the English language (which is shared by European languages but not by Indian languages) is the presence of the indefinite article "a". How does this impact definition structure?
- A common editorial tip for fiction writing is to avoid the overuse of adjectives. For instance, the sentence "he is a very intelligent boy" can be rewritten as "he is a genius". This tip can be couched in property theoretic terms as: "do not supply the context space of the property when

it is not needed". Is this property theoretic reformulation accurate? What is the significance of this tip?

3.2. **Identifying single sentence definitions.** A sentence like "people who do not listen to me are idiots" is not a definition. But the same sentence becomes a definition if we add the word "termed" and say "people who do not listen to me are termed idiots".

The difference lies in the use of the *definitive keyword* "termed".

A single sentence definition is said to have the **property after context**(defined) or **PAC**(defined) property if the property name occurs after the context space specifier. PAC definitions may have the format context space-property name-meaning or the format context space-meaning-property space. For instance, the definitions of agglutinative language and inchoate verb in the past subsection.

PAC single sentence definitions contain a definitive keyword such as "termed", "named", "called", "said to be", "deemed", "christened", or "labeled" to indicate that a new term is being introduced or defined, as opposed to a statement being made about an existing term. For instance, the sentence "a group is called Abelian if any two elements in it commute" is clearly a definition, whereas the sentence "a group is Abelian if any two elements in it commute" does not appear to be a definition, it seems more a statement of fact.

Thus, looking for definitive keywords is a sure way to zero in on the PAC single sentence definitions.

What remain are definitions of the format property name-context space-meaning, that is, sentences where the property is mentioned upfront. Definitions of the format property name-context space-meaning are termed **Context After Property**(defined). This is abbreviated as **CAP**(defined).

For instance, the sentence – "a compact space is a topological space where every open cover has a finite subcover". Or, the definition of voluptuary and superconductor in the last subsection. CAP definitions simultaneously contain:

- An *instance phrase* such as "is a". For instance, when we say "A compact space is a topological space" we are saying that compact spaces are instances over the context of topological spaces.
- A *qualifying phrase* such as "such that" or "where" or "in which" or "under the conditions".

The trouble with CAP definitions is that the format is not unique to definitions. It is not always clear by inspecting a sentence whether it constitutes a CAP definition or a descriptive detail. For instance: "elves are creatures who are bound to work for their human masters" may or may not constitute a definition of an elf.

Here are some pointers that indicate definitiveness:

- The use of indefinite articles
- The degree to which the qualifying phrase is binding. For instance, the phrase "such that" is far more binding than the phrase "where".

Points to Ponder

- Suppose a passage is in print in front of you, as plain text, with no highlighting or emphasis. How will you quickly locate all the definitions?
- Suppose a passage is given as an electronic file (text or print) with word search facilities. How will you quickly locate all the definitions?

3.3. **Statements that confuse with CAP definitions.** Consider the following:

- Blood is the glue that holds together people of a family.
- A dog is an animal that is very easy to domesticate.
- Doctors are people who serve humanity and the community.

None of these are *definitions*. The first of these – "blood is the glue that holds together people of a family" is a **metaphor**. A comparison of two different objects or themes without the explicit use of comparison words, is termed a metaphor. The surface structure of a metaphor makes it appear that the two themes are being equated, whereas, in reality, they are only being compared.

The second and third sentences are not definitions, but *facets of information*.

We must learn how to sieve out the definitions from the metaphors and the facets of information. This requires us to understand the other sentence structures as well. We shall return to the issue in a later tutorial. For now, we must be cautious before identifying a CAP definition.

4. Subtleties in context space identification

4.1. **Relations over a base context space.** Sometimes a property is over pairs of elements in a context space. In fact, *comparatives*, *similarities*, and other *relations* are over pairs of elements. Here are some examples:

- Two **people** are said to be *friendly* if (they get along well with each other).
- One **arithmetic calculator** is said to be *faster* than the **other** if (it takes less time than the other to perform the same set of arithmetic operations).
- Two **triangles** that (have the same angles) are said to be *similar*.

Here, the properties of being *friendly*, *faster*, and *similar* are properties over the context space of – *pairs of persons*, *ordered pairs of arithmetic calculators*, and *pairs of triangles*. That is, given a pair of persons we can ask whether or not they are friendly. Given an ordered pair of arithmetic calculators we can ask whether or not the first is faster than the second. Given two triangles, we can ask whether or not they are similar.

A property defined over pairs of elements in a given space is said to be a **relation**(defined) over that space. A relation over a space is thus a property over the context space over ordered pairs from the original space. The original space shall be called the **base context space** and the space of ordered pairs over it shall be called the **actual context space**.

Here are some sentence structures to watch out for:

- Two `[base context space specifier]` are said to be/termed/called `[property (relation) name]` (optional: to/with each other) if `[meaning]`.

  Here, the actual context space is the space of pairs of elements over the original context space. The above sentence structure is indicative of *unordered pairs*. Mathematicians call such a relation symmetric. A relation is said to be **symmetric**(defined) if the order of the elements being related does not matter.
- A `[base context space specifier]` is said to be/termed/called `[property name (comparative form)]` than/(compared to) another if `[meaning]`.

The above definitions are in the context space-property name-meaning format. The definitions can also be couched in the context space-meaning-property name format. Here are some other tips on identifying definitions of relations (only for the more mathematically minded!!):

- Sometimes, the numeric *two* is omitted. This is usually done in the case of equivalence relations, though it may be done even otherwise. For instance, we may define similarity as "triangles that have the same angles are similar".
- If the meaning begins with "they have the same" or "they are the same" then the relation is an equivalence relation.
- If the meaning does not explicitly say which is which, that is, which is the first object and which the second object from the context space, then the relation is symmetric.
- If the property name is of a comparative form, then the relation is likely to be a partial order, or at any rate, a quasi order. However, this does *not* indicate that the standard form of the adjective connotes a property. For instance, the comparative *faster* may denote a relation for machines, whereas *fast* in itself may not denote a property in our sense.
- If *than* is used after the property name, or in the meaning, the relation is likely to be a partial order or a quasi order.

We can define properties not just over pairs, but also over n tuples, that is, ordered collections of n elements from the space. We can also define properties over unordered collections, or simply subsets, of the space.

Concept Testers

(1) In the original list of definitions(section 3.1), were there any definitions of relations?
(2) Rewrite the above definitions of properties that are relations, varying the property form as property noun and member noun. For instance, the adjective form "similar" gives the property noun "similarity" and the locative form "similarity class".

4.2. **Apparent and auxiliary context space.** In the previous subsection we looked at properties defined over the space of pairs of elements from a given space. So, we talked of ordered pairs of arithmetic calculators, pairs of triangles, and so on. Some properties describe relations between objects from different context spaces. Thus, these properties are over pairs of elements, one from one context space, and the other from another.

Here are some examples:

- The **weather** is said to be *suitable* for an **individual** if (he/she enjoys it and it does good to his/her mind, body and soul).
- The **adapter** is termed *compatible* with the **device** if (the power output generated by the adapter is within the power range needed by the device).

In these examples:

- The **apparent context space** for the property of suitability is the space of possible weather states, and the apparent context space for the property of compatibility is the space of all adapters.
- However, this is not the **actual context space**. Given a weather state, it does not make sense to ask whether or not it is suitable. Given an adapter, it is meaningless to ask whether or not it is compatible.
- Suitability as a property can be evaluated when both the weather state and the individual are given. The actual context space for the property of suitability is thus the space of pairs of weather state and individual.
- Compatibility as a property can be evaluated when both the adapter and the device are given. The actual context space for the property of compatibility is the space of pairs of adapter and device.
- The component missing between the apparent context space and the actual context space is termed the **auxiliary context space**. For the property of suitability, the apparent context space is the space of all possible weather states, and the auxiliary context space is the space of all individuals. The actual context space is the space of all pairs of these. Similarly, for the property of compatibility, the apparent context space is the space of all adapters, the auxiliary context space is the space of all devices, and the actual context space is the space of all pairs of these.

Upshot. When scanning the definition of a property, we should be careful about whether its actual context space is the same as its base context space or apparent context space. It may often be the space of ordered pairs, tuples, subsets instead, or a product with another context space.

So this raises the question: how do we figure out, from the definition structure, whether the apparent context space is the actual context space or not? Here are some sentence structure forms:

- A `[apparent context space specifier]` is said to be/termed/called [property name] in/for/to/with respect to/(some other preposition) a `[auxiliary context space specifier]` if `[meaning]`.

Concept Testers

(1) Identify the apparent context space, auxiliary context space and actual context space in the following definitions:
  (a) A boy is said to be *infatuated* with a girl if he feels madly in love with her.
  (b) A customer calls a service *impeccable* if it meets her expectations completely on all counts.

Points to Ponder

- Sometimes, the auxiliary context space is *implied*, when the agent shirks responsibility. This makes the context space appear narrower. For instance, "a building is called beautiful if the beholder likes it", is a property of a pair of building and beholder, with the space of possible beholders forming the auxiliary context space.

  What is the effect of hiding auxiliary context spaces? How can we locate hidden and suppressed auxiliary context spaces and how can this be used to clarify understanding and bring hidden assumptions to the open?

4.3. **Augmentation of the context space.** Consider the definitions:

- A green triangle is a triangle whose sides are coloured green.
- A compact group is a group whose associated topological space is compact.

In the first example, it is tempting to think that "green" is an adjective describing a property over the context space of triangles. But it does not pass the basic test of a property: given a triangle, it makes no sense to ask *whether or not* it is green.

What is *really* happening is that the adjective "green" denotes a property, not over the context space of triangles, but over a context space of triangles with some additional structure. For instance, we may define a **coloured triangle** as a triangle along with the additional structure of a colour associated with each of its sides. Then, greenness makes sense as a property over the context space of coloured triangles.

We made a similar point in the previous subsection, where we pointed out that auxiliary context spaces may be involved. Here, we observe that the actual context space may involve the apparent context space along with additional structure and the property may make sense only when that additional structure is factored in. The introduction of new structure into the context space is termed **augmentation** and the new context space is termed the **augmented context space**.

In the second definition, things are not so clear except to a person who already can make some sense of the terms! Hence, it is even more likely that a learner (who is not very familiar with the subject matter) will fall in the trap of not realizing the augmentation on the context space.

The structure of the sentence tempts us to think of compactness as a property over the context space of groups. But the meaning given, viz the statement that the "associated topological space is compact", does not make sense in general for a group. It makes sense for an augmented structure – the so called "topological group".

These two examples show that *knowing the previous context spaces and their meanings thoroughly* helps us to identify that the property is not over the apparent context space but over the augmented context space.

There are two steps to understanding such property definitions:

- Identifying and understanding the augmented context space, that is, the additional structure (augmentation) imposed over the base context space. If this augmented context space is already defined explicitly, then first interpreting that definition, otherwise constructing an explicit definition for it.
- Understanding the property as a property of the augmented context space.

We shall come back with a more detailed "how to" on the first count when we study definitions of context spaces themselves, in section 6.

5. Property variants

5.1. **Metaproperties.** A **metaproperty**(defined) is a property of a property. That is, a metaproperty is defined as a property whose context space is itself a property space. That is, given a property over a certain context space, and a metaproperty over that context space, we can ask the question: "does the property satisfy the metaproperty?"

Here are some definitions of metaproperties:

(1) A **property of people** is said to be *hereditary* if (the parent having the property increases the probability of the child having the property).
(2) A **property of formal languages** is called *monotone* if (every formal language that contains a formal language satisfying the property, also satisfies the property).
(3) A **graph property** is called *hereditary* if (every induced subgraph of a graph having the property also has the property).
(4) A *contagious property* is a **property of persons (in their current state)** such that (if a person having the property comes in contact with another person who may not have the property, the other person has an increased chance of acquiring the property).
(5) A **subgroup property** such that (any subgroup having the property in a subgroup having the property also has the original property in the whole group), is called *transitive*.

An examination of the above definitions indicates that the format for defining metaproperties is pretty much the same as for properties, with only one modification. The context space for the metaproperty is now itself a property space – and this fact is indicated by suffixing its context space with the word "property". For instance, "transitive" is a metaproperty for subgroups. The context space of "transitive" is the property space over the context space of subgroups. So, for the context space specifier, we say "subgroup property" or "property of subgroups".

Metaproperties involve two layers of abstraction, and to that extent, they are quite rare. However, identification of metaproperties adds to the toolkit with which we understand and perceive properties. For instance, having encountered the metaproperty of "hereditary", we can ask ourselves each time we encounter a new property: "is this hereditary?"

Metaproperty definitions, if they are given, are usually easy to locate, because the defining sentence itself uses the word "property" or some synonym thereof. In many situations, however, metaproperties are not defined explicitly, but are defined *en passant*[1]. We shall come to the issue of *en passant* definitions in a later tutorial.

5.2. **Property modifiers.** Consider the following sentences:

- I am *very* tired.
- This is *almost* complete.
- The picture is *locally* pretty but *globally* ugly.

Words like "somewhat", "nearly", "almost", "extremely" play the role of *modifying* adjectives. The English language calls these **adverbs** – they are adjectives of adjectives.

The property of being "very tired" is a *modification* of the property of being "tired". The property of being "locally solvable" is a *modification* of the property of being "solvable". These adverbs play the role of *modifying* existing properties, to produce somewhat different properties. Thus, they are termed **property modifiers**.

A **property modifier**(defined) is a rule by which a property over some context space is taken, modified somewhat, and gives a property over the same (or perhaps a somewhat different) context space. Property modifiers are of two types:

- **White box modifiers**(defined) are modifiers whose action depends on the property that they modify. They cannot be applied to any arbitrary property over the context space. White box modifiers usually have no meaning of their own, and only make sense when used to modify the properties they are intended to modify.
- **Black box modifiers**(defined) are modifiers whose action is defined in a general way, so that they can be made to act on any arbitrary property over the context space. Black box modifiers have a meaning (and a definition) of their own, and can be fed in any property.

Black box modifiers are difficult to observe and explain using natural language. Before we discuss *why*, here are a couple of examples:

- Suppose π is a property of graphs. Then, a graph property is termed *hereditarily* π if every subgraph of the graph has property π.
- Let θ be a property of emotional states of persons. Then an emotional state is said to be *irretrievably* θ if it satisfies θ and will remain stuck on θ whatever be done.

5.3. **Explicit definitions of black box modifiers.** For the mathematically oriented, a black box property modifier can be viewed as a function from the property space over a given context space, to itself. In other words, a black box modifier eats a property and spits out another property in the same context space.

In other words, property modifiers are functions, they are machines. A definition of a function must describe how it acts to transform a given input and produce an output.

Defining a function or a map is always tricky with words. The problem is with *repeated referencing.* For instance, try expressing the function $x \mapsto x^2 + 5x + 3$ in words! It will come out as: "the map taking an element to its square plus five time itself plus 3".

Now try the map $(x, y) \mapsto x^3 - y^2 + xy$. In words, this would be – "the map taking an ordered pair of reals to the cube of the first minus the square of the second plus the product of both".

Natural language *can* tackle defining functions, but it is extremely tedious. Why? Because natural language is not designed to repeatedly refer to and recall multiple items. In the symbolic definition $x \mapsto x^2 + 5x + 3$, the letter x coming each time referred back to the original input letter. In natural language, we need to make use of *pronouns* and *positional words* like "first" and "second" and "earlier" and "later". And too much complexity in the function would make the corresponding construction in natural language very tedious. Simply speaking, natural language was not designed for arbitrary levels of complexity, it was designed only for the complexity that may arise in daily communication.

Not that this is a negative feature of natural language – pronouns are one of the most beautiful inventions of natural language, and the fact that we cannot keep shoving in more and more complexity is a positive sign as that is not what natural language was meant for. But it does mean that to *define* a function, we need to borrow the use of symbols to express unknowns unless we want to complicate our lives to much.

And the same holds for property modifiers. To define a property modifier, we need to describe how it acts on an arbitrary property. Defining this action is easily done when the property is given an explicit symbol, which is what we did in the definitions of the property modifiers "hereditarily" and

[1]in passing

"irretrievably". And when we give the property an explicit symbol, we treat the property as an element in its own right – what property theory is all about.

Consider, for instance, the following definitions of "nearly" in the context space of groups:

> A group is said to *nearly* have a given group property if there is a subgroup of the group having finite index in it, such that that subgroup has that property.

Here is the translation with the use of symbols:

> A group is said to be nearly π if there is a subgroup of the group having finite index in it, having property π.

The use of symbols in definition and description is a much wider issue that shall receive more attention later. The point is that *whether or not we use symbols*, an explicit definition of a property modifier is tantamount to an intuitive acknowledgement of property theory.

5.4. **Definitions of white box modifiers.** When we use white box modifiers, we already know what property is being modified. The definition is thus not of the modifier, but of the final modified property. The general formats look like:

- A `[context space specifier]` is said to be `[white box modifier] [original property name]` if `[meaning]`.
- A `[context space specifier]` such that `[meaning]` is said to be `[white box modifier][original property name]`.
- A `[white box modifier] [original property name]` is a `[context space specifier]` such that `[meaning]`.

The three definition formats are the same as those for property definitions. The only difference is that the property we define is that obtained after applying the modifier.

The meaning part of the definition of the modified property may invoke the original property in some way. If this invocation is in a way that does not require us to figure out the definition of the original property, it is likely that the property modifier can be generalized to a black box modifier. This idea becomes important in later tutorials where we learn how to *manipulate* and play around with definitions and develop our own summaries and understanding of theories.

In earlier subsections, we had seen that when properties are expressed in the adjective form, the corresponding property modifiers are expressed in adverb form. When properties are expressed in the member noun form, property modifiers may be expressed as adjectives, adverbs or prepositions.

Here are some examples:

- A **person** is said to be *practically* `[property modifier (adverb)]` *dead* `[property name (adjective)]` if (he or she cannot communicate with the outside world) `[meaning]`.
- A *near majority ballot outcome* is a **ballot outcome** where a candidate misses the majority by just one vote.
- A **painting** is called a *fake van Gogh* if (it looks like a van Gogh but is not actually one).

White box property modifiers are sometimes glued as prefix morphemes to the words themselves. Common modifier prefixes are:

(1) Partial, almost-but-not-quite types: *pseudo*, *quasi*, *semi*
(2) Behind or beyond: *meta*
(3) Many: *poly*, *multi*
(4) Subordinate to, contained inside: *sub*, *hypo*
(5) Over and above, stronger: *hyper*
(6) In the initial stages, before, as a weaker condition: *para*, *pro*, *pre*
(7) Complementary, supportive: *co*
(8) Opposite, a very bad match: *anti*, *ab*, *mal*

Some of these prefixes are *strengthening*, that is, they take the original property to a stronger property. Some are *weakening*, that is, they take the original property to a weaker property. Some take properties to *similar* properties while some take properties to *opposite* properties.

The way property modifiers operate and the formalisms we can use to understand and predict the operations is a huge subject. More of this shall be discussed in later tutorials.

We shall also, in later tutorials, extend the idea of property modifiers to the general notion of **parameterized properties**, and even further to the idea of **property operators**.

5.5. **What makes them challenging.** Metaproperties involve an additional layer of abstraction over properties, and property modifiers are an even more cloaked abstraction. In fact, without using the setup of property theory, it is *hard to understand and appreciate these*. In fact, the act of defining them is tantamount to recognition of properties as elements in themselves.

For this reason, a lot of instructional material circumvents direct definitions of metaproperties and property modifiers, instead resorting to illustrative examples and hoping that we "get the hang". The abstraction is *left to the learner*.

We shall come to these a little more after we understand a little about *en passant* definition styles and how they are employed and decoded.

6. Context spaces

6.1. **Introduction of context spaces.** A context space itself can be introduced in any of these four ways:

- **From scratch**: This definition may be by directly appealing to our sensory or cognitive knowledge. Few definitions are purely from scratch.
- **Via augmentation**: An existing context space is taken and additional information is put on top of its elements. When this additional information is put, a single element in the original context space may give rise to a number of possible elements in the augmented context space. For instance, we begin with the context space being the set of integers, and then augment it by introducing a *flavour*, which could be either sweet, sour or bitter. Then, the same number, say 7, could now correspond to the sweet seven, the sour seven or the bitter seven.

  It may also happen that for some elements in the original context space, the augmentation cannot be done in any way.
- **Via combination**: Give one or more context space, we can consider the space of all ordered pairs over it. This is what we did earlier when we talked of the apparent and auxiliary context space.
- **Via property restriction**: Here, we start with an existing context space, and restrict to those elements that satisfy some property. This restriction defines a subcollection of the context space, which can now be treated as a new context space. Definitions of context spaces by property restriction are, in fact, simply definitions of properties, interpreted as definitions of new context spaces.

Here are some examples of definitions of augmented context spaces:

- A magma is a set with a binary operation defined on it.
- A phone accompanied by a connection and an Internet account, is called a Phone Package.
- A metric space is a topological space along with a distance function on it satisfying certain conditions.
- A trailed packet is a packet along with a trail describing how it moved around in the network.

In the first definition, the original context space is "set" and the augmentation is the "binary operation". The augmented context space is "magma". In the second definition, the original context space is "phone" and the augmentation is "connection" and "Internet account". The augmented context space in this case is "Phone Package".

The formats used are:

- A `[augmented context space specifier]` is a `[original context space specifier]` along with/provided with/accompanied by/endowed with/augmented with `[augmentation]`.
- A [original context space specifier] (along with/provided with/endowed with/accompanied by) [augmentation] is termed a [augmented context space specifier].

Note how this differs from a property definition. In a property definition, we use *qualifying phrases* such as "if", "such that", "where" to precede the meaning, which comprises some *conditions*. In augmentations, we use phrases like "along with", "accompanied by", or "endowed with" to precede the additional structure, indicating that this additional structure is part of the new context space being defined, rather than a condition.

For instance:

- A *coloured partition*`[augmented context space specifier]` is a **partition of the plane** `[original context space specifier]` along with (a colour associated to each part) `[augmentation]`.
- A *complete man*`[augmented context space specifier]` is a **man** `[original context space specifier]` along with (the suit he wears) `[augmentation]`.

Sometimes, new context spaces are defined by simultaneously providing augmentation and performing property restriction. Property restriction done directly with the augmentation is known as *compatibility restriction*. Here are some examples:

- A *semigroup*`[augmented, restricted context space specifier]` is a **set** `[original context space specifier]` with a **binary operation** `[augmentation]` such that (the binary operation is associative) `[compatibility restriction (meaning)]`.
- An **army**`[original context space specifier]` along with a **huge arsenal of weapons** `[augmentation]` such that (the army is adept at using those weapons) `[compatibility restriction (meaning)]` is called a *well armed army* `[augmented, restricted context space specifier]`.
- A *topological group* `[augmented context space specifier]` is a **group** `[original context space specifier]` along with a **topological space structure** `[augmentation]` such that (the group operations are continuous with respect to the topological space structure) `[compatibility restriction (meaning)]`.

Concept Testers

(1) Consider these two definitions:
  - A magician who has a wand is called an empowered magician.
  - A magician along with his or her wand is called an empowered magician.

  Which of these is an augmentation and which is a property definition?

Points to Ponder

- Look at a cross section of definitions from around you. Spot the situations where, from the definition, it is not clear whether it is an augmentation, a property definition, or both (augmentation followed by compatibility restriction).

  Many definitions whose surface structure appears to be a property restriction are actually augmentations or augmentations combined with compatibility restrictions. How can we figure out that a definition is an augmentation when its surface structure is like that of a property restriction?
- We have now seen adjectives in three distinct roles – as property names, as property modifiers (when the property itself is as a member noun), and as augmenting words. A fourth use to which adjectives are put is *grounding*. These are cases where the noun itself is too wishy washy but introducing the adjective gives it some meaning. Can you think of situations where this fourth use comes up? Are there situations where adjectives are used in a way different from all these four?

6.2. **Introduction of elements over the context space.** In all this discussion about properties, we have managed to obscure one point. Properties are important because given any element over the context space, we can decide whether or not it satisfies the property. To really give meaning to properties, we also need to have ways of defining and introducing *elements* over the context space.

For instance:

- Rahul is the tall man you spoke to yesterday on the phone. (provides a definition for Rahul)
- $\mathbb{C}$ is the field that forms the algebraic closure of $\mathbb{R}$.
- The name of this boy shall be David.

Introducing or defining instances is basically about giving enough information to locate a unique element in the context space. This *enough information* can be treated as a highly restrictive assortment of properties such that there is only one element in the context space satisfying them. Note also that while instance definitions follow the same broad outline as property definitions, they make use of the definite article. Another difference is that condition phrases are somewhat more authoritative. While property definitions employ condition phrases such as "such that", "if" and "where", instance definition use phrases like "that" and "which" that are specific and authoritative and pinpoint to a single element.

Key Point 2. *Elements over the context space can be thought of as atomic properties. An* **atomic property***(defined) is a property that is satisfied by only one element over its context space. The specificity and uniqueness reflects in the choice of words and construction of the defining sentence.*

7. Soaking in the definitions

7.1. **Hammer it into the head.** Somewhere in the beginning, we had said that one of the dangers of definition oriented testing was that rote and memorization could be used to beat the system. We *can* memorize definitions without understanding their structure. But *with* the understanding of structure, we can really get the definitions to sink in the mind, absorb them completely, and perceive them in their entirety.

The idea is to synchronize our *learning mechanism* with the *content and structure* of the definition. Here is the broad strategy for **definition mastery**:

(1) **Identification**: Identify the property name, context space and meaning.
(2) **Formulation**: Express the definition in each of the three formats, and writing the property in as many forms as possible: adjective, property noun, member noun, and so on.
(3) **Rendition**: Render the property exploiting the best learning mechanism.

Adequate background has been developed here for the identification and formulation stages. Here we discuss the rendition stage.

The mind makes use of **representational systems** to grasp ideas. Three representational systems used most extensively are:

- The **visual system**(defined), which uses sight
- The **auditory system**(defined), which uses sound
- The **kinesthetic system**(defined), which uses touch and feel

Rendition is most effective if the powers of all the representational systems is combined in a manner that plays up to the learner's strengths. The coming subsections discuss how this can be achieved.

7.2. **Recitation and incantation.** *Recite* the definition. Speak it aloud. With modulation, intonation, with feeling and meaning.

We shall focus on three aspects of recitation:

- The volume
- The pitch (shrill versus bass)
- The speed

A property definition has three parts – the property name, context space, and meaning. The style of incantation must *reflect the separate natures of these three parts*. Here are the recommended recitation techniques for each of the formats.

(1) The **property name-context space-meaning** format, where we say:
  - The property name loud, shrill and quick. This helps in *highlighting* it.
  - The context space specifier at a medium pace, with the pitch indicating an incomplete and interrogatory tone. This helps reinforce the point that given an element of the context space, the question will be asked as to whether or not it satisfies the property, and the answer to that question lies in the meaning, which is coming up.
  - The meaning in a slow, final, and low pitched voice with adequate gaps between the words. The slowness and lowness is to ensure that every word of the meaning is clear and that it is the final and authoritative word on the property.
(2) The **context space-meaning-property name** format, where we say:
  - The context space in a medium voice, with a slight interrogatory tone to it.
  - The meaning very slowly, in a low pitched voice, but not with finality.
  - The property name in a tone of suppressed triumph, somewhat louder.
(3) The **context space-property name-meaning** format, where we say:
  - The context space slowly and then pause. This helps in emphasizing the context space.
  - The property name with a thrust emphasis, high pitch and fast, with an interrogatory tone. This again helps in highlighting it.
  - The meaning in a slow, final and low pitched voice.

In each of these cases, the roles played by the styles of inflection and modulation are different. So it is best to formulate the definition in all forms and practise each of these. Do not try to go very rigidly by any of the above guidelines – use your own intuition of what the definition is telling and how best to convey it.

As we have seen, terms defined need not always be properties. For each kind of term being defined, the intonation and modulation techniques may vary somewhat. The manner of recitation must bring out the *actual context space* and prevent incorrect assumptions about the context space getting settled in the mind.

Points to Ponder

- Is the recitation successful in reinforcing the fact that the term being defined indicates a property that is binary ,definite, intrinsic and over a context space?

- Think of how to best recite definitions of relations, or properties over ordered pairs or tuples.
- Suppose we had to commit to memory the definition of a metaproperty. In what ways would the recitation procedure differ from that for a property? The difference should be such as to reinforce the fact that this property is being evaluated over properties.
- Suppose we had to commit to memory the definition of a property modifier. What recitation procedure should we use? Go back to some of the examples of property modifiers and try out intonation styles for them.

7.3. **Writing and reading.** One way of practising a definition is to write it and then read it. While writing, we need to ensure the conceptual separation of the property name, the context space, and the meaning. This can be done using any of these tools:

- Using **colour coding** by allocating different colours to the context space specifier, the property name, and the meaning. Associate a fixed colour for the context space specifier, a fixed colour for the property name, and a fixed colour for the meaning, and stick to the same colour convention across all the formulations of the definition.
- Using font or **bold highlighting** or *emphasis*.
- Using a visual representation. One common representation style is the **Venn diagram** style, where the context space is drawn as an oval and the subcollection satisfying the property is drawn as an oval inside that. The meaning is then indicated separately as the *distinguishing factor* between those inside the smaller oval and those inside the rest of the bigger oval.
- Using an equational representation. More on this will emerge in the later tutorials.
- Using indexing shorthands. These shall also be discussed in the later tutorials.

POINTS TO PONDER

- What techniques (among the above) were used in this text? Were they effective? How would you do things differently?

7.4. **Hand gesturing.** While either reading or writing or hearing or speaking the definition, use hand gestures if possible. here are two ways of using hand gestures:

- In a generic fashion. For instance, a particular hand gesture for the context space specifier, a particular hand gesture for the property, and a particular hand gesture for the meaning.
- In a specific fashion, where the particular words used in the definition correspond to particular hand gestures.

7.5. **Mental pacing.** Pace the mind with the rendition process to really get the definition etched in the mind. This is true whatever be the representational system in use. In other words, control the thoughts as the definition is rendered.

There are many pacing strategies. One of them is **general example based pacing**. This proceeds as follows:

- When the context space specifier is rendered (read, written, spoken or heard) run the mind through the meaning and the various instances of the context space. Get a flavour of the context space in the mind.
- When the property name is rendered, do a mental segregation of the instances where the property is held, and the instances where it is not held.
- When the meaning is rendered, *evaluate* this meaning in the instances where the property is held, to get true, and in the instances where the property is not held, to get false.

It may seem difficult to cram all these thought process so quickly. We do not need to thoroughly go through them. What is necessary is that the sensation, the flavour, of those thoughts come into the mind each time the definition is rendered. This helps the mind forge the correct connections and makes it more and more difficult for the definition to slip out of the mind.

Another method is the **symbol based pacing**. It relies on two formulations of the definition: one "verbal' and the other "symbolic". The method is: "render the verbal form using one representational system and the symbolic form using another, simultaneously and synchronously". For instance, consider the following two forms of the definition of Abelian group:

A group is Abelian if any two elements in the group commute.
G Abelian $\equiv ab = ba \ \forall \ a, b \in G$

Now, while reciting the verbal definition, we may move our finger over the corresponding components of the symbolic definition. Or perhaps even mentally move over the corresponding components of the symbolic definition.

7.6. **When and where to revise the definitions.** The definition is best assimilated at times when the mind has the right overall feeling and connections established for it. Here are some favourable points in time:

- Just after managing to understand the definition in terms of its various parts, and getting a rough idea of its meaning.
- After completing a study session where the definition was first encountered, or perhaps at some optimal time on the same day.
- After coming across an example or non example that serves to either increase confidence or reveal misconceptions about the definition.
- On coming across an exercise or rider or "real life situation" that required, or appeared to require, some ideas related to the definition. At this time, it is best to first recall the definition from memory, and then check it against a reliable source.
- While preparing or going through a summary.

8. Some done, more to do

8.1. **So far.** We have figured out that definitions are not just flat, they have structure, they have parts, and they have formats. Understanding the underlying parts helps us commit the definition better to memory. But there is a bigger advantage. When we try to figure out the property name, the context space, the meaning, and all these, we get to better place the definition in the overall framework and our comprehension level improves. Moreover, this reflects in improved performance in both "theory" and "application" testing.

The terms we define may be:

- **Properties** as property nouns, adjectives, member nouns, or locatives. The property must have a definite context space.
- **Relations** or properties over ordered tuples of elements across multiple spaces. Relations are also properties, if we consider the **actual context space** to be the collection of all ordered tuples with elements from the individual context spaces.
- Properties over **implicitly augmented context spaces**.
- **Metaproperties**, which are properties whose context space is the property space over a context space.
- **Context spaces** themselves. Context spaces may be defined from scratch, via augmentation, via combination or via property restriction.
- **Elements** in a given context space. An element can be thought of as an **atomic property**, a property that is true for precisely one element.
- **Property modifiers** which take in a property and modify it somewhat. There are **white box modifiers** and **black box modifiers**. Black box modifiers can act on any property over the context space. Property modifiers occur as adverbs.

There are also **element operators**, **property operators** and **parameterized properties** that we shall see in later tutorials.

So far, we have seen how the use of grammar and sentence structure can help determine what kind of term is being defined, and then identify the corresponding context space and make more sense of the meaning.

8.2. **What is next.** Our treatment of definitions has been weak on many counts. The later tutorials will fill up the gaps. We have looked mainly at single sentence definitions, but a lot of relatively light material relies on gradual build up in definitions. Even technical and precise material uses the single sentence definition format only for terms whose meaning needs to be spelt out precisely, many other terms are just built up via a gradual understanding and consensus. Further tutorials will cover approaches to tackle varying definition styles.

The crux of a definition lies in its *meaning*. We now need to develop *tools to understand the meaning*. Property theory supplies us with many gadgets and formalisms, and in the next tutorial, we familiarize ourselves with those formalisms and equip ourselves better for understanding the meaning.

These formalisms also help us tackle statements other than definitions – for instance, statements of fact such as “All blue haired people are European” or “every characteristic subgroup is normal” or “all diabetics have hyperglycaemic tendencies”.

Incidentally, the formalisms and notations we develop shall also provide learning strategies alluded to in the previous section.

In later tutorials, we shall also see how to tackle reading material, spoken instruction in class, and other forms of instruction more effectively.

Index