GLOBAL CALCULUS: BASIC MOTIVATIONS

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Abstract. This is the first article of a multi part series on global calculus, based on the course covered by Professor S Ramanan in the Chennai Mathematical Institute in August–December 2005. This article focusses on developing the basic motivations for global calculus and the study of differential manifolds, and introduces the relevant ideas from sheaf theory. The article is not entirely faithful to the course and has some extra material and perspective on sheaf theory that was covered in later parts of the course or not covered at all.

1. The raison d'etre

1.1. **Local models, global objects.** The Earth is round. But in a sufficiently small region, it looks flat. In fact, we can make a circular enclosure on the ground, and map this enclosure to a circle in a flat plane. They are "almost" the same.

To make precise what we mean by this, we need to address two questions:

- In what sense are they the same? What geometrical properties do they share?
- Can these local resemblances be used to study the Earth as a *global* object in terms of our understanding of the plane?

Mathematics has come well equipped with certain notions of resemblance. These include:

- Topological **homeomorphism**. Two topological spaces are said to be homeomorphic if there is a bijection between them that is continuous both ways. This map is called a **homeomorphism**. Homeomorphism is the basic notion of topological equivalence.
- **Diffeomorphism**. This basically preserves the structure of "differentiable functions". A diffeomorphism between two spaces, each with a notion of differentiable functions, is a homeomorphism that maps differentiable functions to differentiable functions.
- **Isometry**. When the two spaces in question are metric spaces, that is, they have a notion of distance, then an isometry is a bijection that preserves distances. An isometry gives a homeomorphism on the topology induced by the metric.

Each of these is of independent interest. For our purposes, however, what interests us is the *differentiable functions* that we can define on the space, and *phenomena arising from there.* Thus, diffeomorphisms are of primary interest.

1.2. **Phenomena and language.** Suppose we want to describe a sphere, a torus, or a Mobius strip. But we have with us only one language available: the language of the plane, $\mathbb{R}^2$. Then, we construct *local models* of the sphere via the plane satisfying certain *compatibility conditions.* We now want to study phenomena on the global object. There are two possibilities:

- The phenomena are *intrinsic* to the global object, and are independent of the choice of the local models.
- The phenomena are *specific* to the choice of the local models.

Largely, our interest lies in intrinsic phenomena, because, after all, the object we originally intended to study was the global object. The local models are just tools for our convenience, they are not an end in themselves.

However, at least to begin with, we do not have a language to directly handle the global object. With the course of time, we will develop such a language. As we proceed, we will need to define and introduce phenomena associated with the global objects. This will be done in two ways:

- Intrinsically, viz *directly* in terms of the global object.
- Indirectly, in terms of the local choices. It is then our prerogative to show that the definition, though *a priori* dependent on local choices, is actually independent of them.

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Often, we shall provide *both* kinds of definitions, using the first kind for proving theorems, and using the second kind for computations. Of course, to show independence of local choices, it suffices for us to demonstrate that the definition of the second kind is equivalent to a definition of the first kind.

Points to Ponder

- One of the simplest examples of intrinsic versus extrinsic comes up when we study homomorphisms between vector spaces. Given a suitable choice of basis, we get a bijection between the set of homomorphisms and the set of matrices of the corresponding order. However, the association of the homomorphism to the matrix is not *intrinsic*.

  Given two vector spaces V and W, both of dimension n, is the *determinant* of the associated matrix intrinsic to the linear transformation? What about the trace? What if we look at endomorphisms of a vector space (that is, identify V and W)?
- For the cases identified as intrinsic, is there an intrinsic formulation that does not require choosing a basis at all?

1.3. **Notion of compatibility.** Suppose a translator between languages A and B translates word to word, that is, each word in language A goes to a corresponding word in language B. Suppose two people known only language B, and they both feed in a sentence in language A to the translator. Obviously, they cannot check if the translator has worked correctly. However, they do know that whatever words were common to the two sentences, should have got translated in the same way (or, at least, in equivalent ways) for both. This is the idea behind requiring *compatibility on intersections*.

We want to describe a manifold using the language of local models. Now, the natural language for the differential manifold (which we, as yet, do not understand) is the language of the global object (this is the language A above). But for the purpose of our understanding, we translate it to local models, which are in places we are familiar with (that is, in language B). But in order for this translation to be meaningful and consistent, the same things in language A should translate to equal things in language B.

Enough of background! Now for the main idea. A topological manifold is a structure that is *locally* like a Euclidean space, that is, it is a topological space such that for every point, there is an open neighbourhood of it homeomorphic to an open set in $\mathbb{R}^n$.

So far, so good, because a topological manifold is first and foremost a topological space, and all topological phenomena can be expressed for it. The problem arises when we start trying to *translate* the additional structure on open sets in $\mathbb{R}^n$ – such as differential or analytic structure, to the manifold. As in the A to B language translator above, we have no way of directly commenting how the translation will look. What we *can* say is that the translation must be in a *compatible* fashion.

The question is: when the topological manifold does not have a differential structure from before, how can we transfer the structure of differentiable functions from $\mathbb{R}^n$ to it?

Let us try to suppose what the consequences would be if it *were* a diffeomorphism. The situation is: To each open set in the topological space, we associate an open set in $\mathbb{R}^n$. Now, if this were a diffeomorphism, then consider the following: take two open sets in the topological space, look at the corresponding open sets in $\mathbb{R}^n$, and the corresponding homeomorphisms. Now, look at the intersection of the two open sets. This maps homeomorphically to a subset of both the open sets in $\mathbb{R}^n$. And we would like each of those maps to be a diffeomorphism. This means that we require that the induced map between the open subsets of $\mathbb{R}^n$ is a diffeomorphism.

This is the condition of **compatibility on intersections**(explained), that is, the induced homeomorphism on the intersection of open sets between the corresponding sets in $\mathbb{R}^n$, is in fact a diffeomorphism.

Key Point 1. *To impose an additional structure via local models, we enforce compatibility on intersections, in the sense that the induced map is an equivalence with respect to that additional structure.*

The trouble with the above discussion is that it involves a lot of hand waving, and it seems extremely confusing!! In the next section, we shall introduce the concept of **sheaf**(first used). This concept is the first step towards understanding how to give a **global datum** for an object. But before that, let us salivate a little more with some glimpses of what we are going to do.

1.4. **What we do in calculus.** One of the basic goals of calculus is to understand *local behaviour* and *local change*. This is not the same as *point wise behaviour*. When we talk of point wise behaviour of a function, we are referring to its values at a single point. When we talk of local behaviour, we refer to its value in a *sufficiently small neighbourhood* of the point. That is, we would like to say: "In some sufficiently small neighbourhood of the point, the function behaves so and so".

Thus, our aim is to get a handle on local behaviour. And the first and most common handle on local behaviour is the "derivatives". Knowing the derivatives of a function at a point gives us insight into local behaviour in the neighbourhood of the point. As shall turn out to be crucial, this insight is far from complete.

Many laws of nature operate at the local level, and most of them express relations between the functions and their instantaneous rates of change. From these laws, we are required to determine the global behaviour. And the space on which the body is constrained to be is a manifold. Thus, global calculus creates the appropriate language for us to figure out what is going on.

2. Presheaf: first round of definitions

2.1. **A rough preview.** We develop here enough of the theory of sheaves and presheaves to be able to understand smoothly the translation between the global structure of a differential manifold and its local models.

The notions of sheaf and presheaf are largely functorial, in the sense that given any category, such as the category of groups, of rings, of modules, of algebras, we can talk of a "sheaf (presheaf) with values in the category". In most applications, the starting categories are quite limited. There are many ways of viewing and developing the theories of sheaves and presheaves:

- One way is to look at sheaves and presheaves over concrete categories, and then, via the forgetful functor to sets, view them as sheaves and presheaves over the category of sets. This approach helps describe many phenomena and properties at the set theoretic level. These are **set theoretic phenomena**(defined) and make sense only for presheaves or sheaves over concrete categories.
- Another approach is to look at general phenomena over the category of sheaves and presheaves. These are called **category theoretic phenomena**(defined).
- Yet another approach is to extend phenomena from the original category to the sheaf or presheaf category. Phenomena of this kind are called **value theoretic phenomena**(defined).

Some of the material that we study is not relevant to the development of global calculus at this juncture. It has been included to ease the introduction of new ideas in the later parts.

2.2. **Presheaves.**

Definition. Given a category, a **presheaf**(defined) with *values in* the category is a topological space equipped with the following data:

- To every open set of the topological space, associate an object of the category.
- To every inclusion of one open set in another, associate a morphism from the object corresponding to the bigger open set to the object corresponding to the smaller open set. The morphisms should commute and the identity inclusion should induce the identity morphism. The morphisms associated with the inclusion map are called **restriction maps**(defined). The reason for this terminology shall be clear soon.

New Notation 1. Presheaves are typically denoted by capital calligraphic letters like $\mathcal{A}$, $\mathcal{O}$. The underlying topological space is denoted by capital English letters such as M, X, Y (the letter M is typically used when the topological space is a manifold). Points in the topological space are denoted by small English letters such as m, p, x and open sets are denoted by capital English letters like U, V and W.

The presheaf $\mathcal{A}$ on a topological space X is sometimes written as $(X, \mathcal{A})$. The object associated with an open set U of a space X for a presheaf $\mathcal{A}$ on X is denoted as $\mathcal{A}(U)$. It is also called the *value* taken at U.

For presheaves taking values in a concrete category, elements in the object corresponding to an open set (that is, members of $\mathcal{A}(U)$) are denoted typically by small English letters such as f, g. These letters are chosen because these elements are often functions.

The restriction map from an open set U to an open set $V \subseteq U$ is denoted as res_{UV}. Thus, the restriction of $f \in \mathcal{A}(U)$ to $\mathcal{A}(V)$ is denoted as $\text{res}_{UV}(f)$.

Here's another, somewhat more technical definition:

Alternative Definition. Given a topological space, define its *open set category* as the category whose objects are its open sets and whose morphisms are the inclusion maps of open sets. A presheaf with values in a category is then a topological space equipped with a contravariant functor from its open set category to the given category.

We shall see, in a later section, that presheaves come under a more general notion called a **coefficient system**(first used).

Let us begin with the category of sets. Suppose M is the topological space and S is a fixed set. Given any subset U of M (open or not), we can consider the set S^U of functions from U to S. Define, now, a presheaf $\mathcal{A}$ as follows:

- For every open set U, associate the *set* of functions from U to S. That is $\mathcal{A}(U) = S^U$.
- If $V \subseteq U$ define the **restriction map** from S^U to S^V simply as function restriction, that is, given a function from U to S, *restrict* it to V to obtain a function from V to S.

The restriction maps clearly satisfy the conditions required, that is:

- Restrictions commute, that is, if $W \subseteq V \subseteq U$ then restriction from U to V and then to W is the same as direct restriction from U to W.
- The identity inclusion gives rise to the identity restriction.

Instead of looking at all functions, we can give S a topology as well and look at the *continuous* functions from open sets to S. Here, instead of looking at *all* functions, we are looking at a subset of the set of all functions. This motivates a little definition.

Definition. A **subpresheaf**(defined) of a presheaf (with values in a concrete category) is another presheaf such that:

- The underlying topological space is the same.
- The category in which values are taken is the same.
- For every open set, there is (given) an injective homomorphism from the value taken in the (alleged) subpresheaf to the value taken in the alleged presheaf, such that these injective homomorphisms commute with restriction maps. In other words, on each open set, it can be viewed as a subobject in a consistent way.

In symbols, suppose $\mathcal{A}$ and $\mathcal{O}$ are two presheaves on a topological space X with values in a given concrete category. Then , to talk of $\mathcal{O}$ as a subpresheaf of $\mathcal{A}$, we need to, for every U open in X given an injective homomorphism σ_U from $\mathcal{O}(U)$ to $\mathcal{A}(U)$ such that if $V \subseteq U$, then the composite of the restriction map in $\mathcal{A}$ from U to V with σ_U is the same as the composite of σ_V with the restriction map in $\mathcal{A}$ from U to V.

For instance, the presheaf that associates to each open set the set of continuous functions from it to a fixed set S, is naturally a subpresheaf of the presheaf that associates to each open set the set of all functions from it to S. The injective homomorphism in this case is simply the map taking each continuous function to itself (now viewed as a function). Clearly, this commutes with restriction maps, because restricting as a function is the same as restricting as a continuous function.

The above definition is *value theoretic* because it refers to relations between the values taken at the various open sets, using notions defined in the original category.

2.3. Homomorphisms of presheaves.

Definition. The presheaves with values in a given category and over a given topological space themselves form a category. A **homomorphism**(defined) from one presheaf to the other and a rule that, to each open set in the topological space, gives a homomorphism from the corresponding object in the presheaf, to the corresponding object in the second presheaf, such that the homomorphisms commute with restriction maps.

Thus, if X is a topological space, and $\mathcal{A}$ and $\mathcal{O}$ are two presheaves over X, then $\sigma : \mathcal{A} \to \mathcal{O}$ associates, to each open set U in X, a homomorphism $\sigma_U : \mathcal{A}(U) \to \mathcal{O}(U)$ such that $\sigma_U.\text{res}_{VU}\mathcal{A} = \text{res}_{VU}\mathcal{O}.\sigma_V$ whenever $U \subseteq V$ are open sets.

The above definition gives a new perspective to subpresheaf: A presheaf is a subpresheaf of another if there is a presheaf homomorphism that is injective on each open set.

Note that there is a funny functorial feel about this. A homomorphism of presheaves over a given category associates, to each open set, a homomorphism. So, can this be made a presheaf as well?

Given a category, we can define a concrete category, the **homomorphism category**(defined) of the original category, as follows:

- Its objects are the sets of homomorphisms between pairs of objects in the original category.
- A homomorphism of this category is given by providing a homomorphism of the left object and the right object.

We'll be particularly interested in analyzing the nature of the homomorphism category of the category of presheaves with values in a given category. (This may seem a mouthful at the moment, but it does come up quite often, and is not as contrived as it looks!)

2.4. **Direct image of a presheaf.**

Definition. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{A}$ a presheaf on X. The **direct image**(defined) of $\mathcal{A}$ under f, denoted as $f\mathcal{A}$, is the presheaf on Y that associates to each open set U in Y, the object $\mathcal{A}(f^{-1}(U))$.

Continuity of the map is used in the fact that the inverse image of every open set is open. We shall later on explore the ideas of inverse image and f cohomomorphisms, after studying sheaves.

3. Functions: our primary motivation

3.1. **Some notions pertaining to functions.** In this section, we develop a few ideas for the study of functions on a space. By *functions on a space* I mean continuous functions from that space to $\mathbb{R}$ or perhaps $\mathbb{C}$. There may be situations where we look at functions to other spaces as well.

The idea, as remarked earlier, is to use local models in the global study of a topological space. One way of quantitatively approaching these local models is by defining function related notions on them.

In the next section, we shall generalize these notions to arbitrary presheaves.

3.2. **Germs, stalks and etale spaces.** Let's look at two topological spaces X and Y, and a point x in X that we are interested in. We would like those properties of a function that depend only on its value quite close to x. Thus, two functions that coincide in a sufficiently small neighbourhood of the point should be considered equivalent. This gives an equivalence relation for each point:

Definition. Given two functions f and g defined in open neighbourhoods U_1 and U_2 of a point x in a topological space X, we say that $f \sim g$ at x or that f and g *have the same germ* about x, if there is a neighbourhood of x contained inside both U_1 and U_2, such that $f(y) = g(y)$ for every point y in that neighbourhood.

That the relation is reflexive and symmetric is direct from the definition. Transitivity follows from the fact that a finite intersection of open sets is open. The equivalence classes under this equivalence relation are called **germs**(first used) of functions at the given point. The set of all germs of functions at a given point (that is, the quotient space under the equivalence relation) is called the **stalk**(first used) at the point and the disjoint union of all these stalks is termed the **etale space**(first used). The etale space also has a topology, that we shall come to soon. Then, it will become clearer just *why* germs, stalks and etale spaces are important.

Visual Analogy 1. *Imagine that the topological space from which functions are being considered is lying flat on the ground, with each point being a point on the ground. The* stalk *at each point is a vertical stalk, like a stalk of a plant growing from that point in the ground. And the union of all these stalks is the etale space (note that no two stalks intersect, because they are all straight upwards and they all have different points on the ground).*

Each point on the stalk of a point is a germ at that point. The union of all the stalks, which is the entire plant, is called the etale space.

The above definition makes sense for functions from any topological space to anywhere.

A phenomenon associated with a function with respect to a point is said to be **local** if it depends only on the germ of the function at that point. As such, a property of a function is local if it can be broken down into properties of its germs at the various points. Thus, the derivatives are local, and the property of being continuous, differentiable, and so on, are all local. The property of being **bounded** or being **integrable** are not local.

3.3. **Functions to sections of the etale space.** Suppose we are looking at the set of all functions from a set S to a set T. One way of understanding them is to look at them as subsets of $S \times T$ such that the projection map to S is bijective (that is, every element of S has exactly one lift). Thus, a function from S to T can also be thought of as an *inverse* to the projection map from $S \times T$ to S. Such an inverse is called a *section*, because we are basically taking a kind of cross section of $S \times T$, representing each element of S exactly once.

When doing calculus and topology, we continue with the same idea, but now, we try to encode data via which we can relate properties of the functions that we started with, to properties of the corresponding sections.

Key Point 2. *Many of the idea in topology and calculus dealing with continuous and differentiable functions are simply ideas dealing with functions between sets. There is, however, an added responsibility of ensuring that the new topological and differential data is maintained and transferred.*

The naive way of extending this idea to continuous functions from a topological space M to $\mathbb{R}$ is as follows: take the product space $M \times \mathbb{R}$, and look at sections from M to this. The projection from $M \times \mathbb{R}$ to M has, for each point, simply a copy of $\mathbb{R}$ as its inverse image. The value that we choose at the point is simply the value taken by the function. Thus, this kind of construction would be good at capturing **point properties** of functions.[1]

Instead, if we want to study *local properties*, then, to each point, we must associate not just the *value* that the function takes, but the *germ*. But this means envisaging a space and a projection map so that the inverse image of each point is the *stalk* at that point. And the space in question is just the *etale space*.

Visual Analogy 2. *The stalks of the etale space are too huge. Each point has a very huge stalk. And only a very small part of this stalk is needed to calculate the value of the function at that point. If we take that* much smaller stalk like thing*, and take the union of those, we get an overall space that is significantly smaller than the etale space.*

Notice that the etale space is *much bigger* than the space obtained by taking a copy of $\mathbb{R}$ at each point. The difference lies because the etale space gives local behaviour at each point, as opposed to pointwise behaviour.

So far, we have not used anything about the algebraic structure of $\mathbb{R}$, but now let us say something. $\mathbb{R}$ is a field, and the set of germs of all functions at a point forms a *ring*. This is because the equivalence relation of *having the same germ* is a congruence with respect to the ring structure of all continuous functions. So, the set of germs is a ring.

The ring is an $\mathbb{R}$ vector space. Even better: consider the ideal of germs that take the value 0 at the given point. This ideal is a maximal ideal, and the quotient by this ideal is very naturally $\mathbb{R}$ – every coset of this ideal corresponds to a particular value taken by the function at the point. Thus, the copy of $\mathbb{R}$ at every point in one space, arises as a quotient by a maximal ideal of the stalk in the etale space.

Thus, the space with a copy of $\mathbb{R}$ at every point in the manifold is a *much smaller* space than the etale space – while the etale space encodes the germs at all points, this space only encodes the values at all points.

In effect:

- Given a function from a topological space X to another topological space Y, we can express it as a section of the projection map from their Cartesian product $X \times Y$ (as a set) to X. The section takes each point in X to the value taken by the function in Y. We are interested in the case $Y = \mathbb{R}$.
- The function from the topological space X to the topological space Y can also be viewed as a section of the projection map from the etale space to X. The section takes each point in X to the corresponding germ of the function in its stalk. We are interested in the case $Y = \mathbb{R}$.

Functions may not be defined on the whole space, they may be defined on open sets. This leads to the following somewhat modified pair of observations:

- Given a function from an open set U to a topological space X to another topological space Y, we can express it as a section of the projection map from their Cartesian product $X \times Y$ (as a set) to X, restricted to the open set U. The section takes each point in U to the value taken by the function in Y.
- The function from the open set U in the topological space X to the topological space Y can also be viewed as a section of the projection map from the etale space to X, again restricted to U. The section takes each point in U to the corresponding germ of the function in its stalk.

Visual Analogy 3. *The projection map from the etale space gives, for each germ at a point, just the point. In other words, given a germ, which is a point on the stalk, it projects to the point on the ground directly below it.*

[1]The space with a copy of $\mathbb{R}$ at each point is isomorphic as a set to $M \times \mathbb{R}$ but it may not have the product topology. All we know for sure is that the projection map is a fiber map.

A section to this map gives, for each point in the ground, a point in the stalk (that is, a germ).

3.4. **A topology to the etale space.** It is clear that every function from one topological space to another gives a section of the etale space, but is the converse true? It does not seem so, because a section of the etale space will give local information about each point – and this local information may be conflicting. So a direct correspondence is not very promising. However, we can try to restrict our attention to *continuous* functions between the topological spaces to begin with, and *continuous sections* of the etale space. To do this, we need to define a suitable topology on the etale space.

We would like to call two points in the etale space close only if their projections are close. But, further, we would also want that the germs they correspond to, are fairly close in nature, and can be lifted to similar looking functions.

Visual Analogy 4. *Recall the picture of the topological space being flat on the ground and the stalks each being vertical, rising from the points on the ground. Two points in two stalks are* close *if their ground points are close, and if they also have a similar height, which means, in other words, that they are germs of a common function.*

Here, then, is the topology. Let X be the starting topological space, Y the image space (usually $\mathbb{R}$ or $\mathbb{C}$), and $\mathcal{E}$ be the etale space. To each open set U in X, and every continuous function f from this open set U to the other topological space Y (usually $\mathbb{R}$ or $\mathbb{C}$) associate the following set in $\mathcal{E}$: the set of all germs of f at points in U. Use this as a basis.

We need to convince ourselves that this is indeed a topology. Indeed, all we need to check is that the intersection of two basis elements is a union of basis elements. If (U, f) and (V, g) are two basis elements, then the intersection is the set of points in $U \cap V$ where the germ of f is the same as that of g, along with that germ. But if the germ is the same at a point, there is a neighbourhood about that point where the germ is the same. This provides a cover of the intersection with basis elements.

We also need to check that the projection map is a continuous map with respect to the topologies given. Under the projection map from $\mathcal{E}$ to X, the inverse image of any open set U in X is the set of all elements of $\mathcal{E}$ comprising points of X and germs of functions defined in neighbourhoods of those points. But this is a union of open sets – given by taking, for every point $x \in U$ and every germ about that point, an open set $V \subseteq U$ and a function f over V such that the germ of f at the point is the required germ.

So far, so good. Next, we have to figure out what a *continuous section on an open set* means. Let U be open in X and s be a section from U to $\mathcal{E}$, that is, $s(x)$ is in the stalk of x for each $x \in U$. What does it mean if s is continuous? The pre-image of every open set must be open. Thus, if (V, f) gives a basis open set in $\mathcal{E}$, then $s^{-1}(V, f)$ must be open in X.

Now, $s^{-1}(V, f)$ is precisely those points x in U such that $s(x)$ corresponds to a germ of f. This being open means that there is an open set in U such that the restriction of f to that open set gives rise to the section s, restricted to that open set.

The above idea can be used to generate an open cover of U such that:

- For each open set in the covering, there is a function such that the corresponding section is s restricted to that open set.
- The functions corresponding to any two open sets agree on their intersections.

Thus, a continuous section over U gives rise to a *local description* of a function in U, via descriptions of functions on an open cover of U. It is clear that because continuity is a **local property**, the local description can be pieced together to give a continuous function on U, in a unique way. Thus:

Upshot. Continuous functions from open sets in X to Y correspond to continuous sections from open sets in X to $\mathcal{E}$, of the projection map from $\mathcal{E}$ to X (where $\mathcal{E}$ is given the topology of the etale space described here).

4. Sheaf: second round of definitions

4.1. **Germs, stalks and etale space for presheaves.** We had defined germs, stalks and etale spaces for functions. The collection of continuous functions can be thought of as a presheaf – to each open set, associate the continuous functions defined on it. Then two functions are equivalent if their restrictions to some smaller open set are identical.

In the general language of presheaves, we have a general concept of *restriction*, which, for presheaves of functions, has been interpreted in terms of function restriction. In fact, for a presheaf taking values in a concrete category, we can define the notions of germ, stalk and etale space. These notions can be defined *set theoretically*, as we see below.

Definition. Consider a presheaf with values in a concrete category. Consider pairs of the form (U, f) where U is an open set in the topological space and f is an element in the object corresponding to U. Then we have $(U, f) \sim (V, g)$, or we say that (U, f) and (V, g) have the same **germ**(defined), if there is a W in $U \cap V$ such that the restriction from U to W of f is the same as the restriction from V to W of g.

We now repeat some definitions:

Definition. The germ, stalk and etale space for a general presheaf:

- A **germ**(defined) at a point is an equivalence class of elements defined corresponding to open neighbourhoods of that point, under the equivalence relation of *having the same germ*. Thus, the germ at a point $x \in X$ for a presheaf $\mathcal{A}$ over X associates, is an equivalence class of elements of the form (U, f) where $x \in U$, U is open in X, and $f \in \mathcal{A}(U)$, under the equivalence relation of having the same germ.
- The **stalk**(defined) at a point is the set of all germs at the point. In other words, it is the quotient of the set of all elements over open neighbourhoods of the point, by the equivalence relation of being in the same neighbourhood.
- The **etale space**(defined) is the disjoint union of all stalks.

New Notation 2. For a presheaf $\mathcal{A}$, the stalk at a point x is denoted $\mathcal{A}_x$. The etale space is the disjoint union of stalks and is denoted as $\sqcup_{x \in X} \mathcal{A}_x$.

4.2. **A comment on stalks.** We know that the set of continuous functions (to $\mathbb{R}$) on any open set form a ring under pointwise addition and pointwise multiplication. Thus, the presheaf of continuous functions can be viewed as a presheaf of rings. We also know that the stalk at any point is also a ring.

Thus, the stalks are also *objects of the category* over which we are taking the presheaf.

We defined the notion of germ for a general presheaf. The set of germs was termed the stalk. However, in general, it is not clear whether the stalk can naturally be given the structure of an object in the concrete category over which the presheaf is being considered.

As it happens, for most algebraic structures, it can. The only condition required is that the equivalence relation of *having the same germ* is a congruence with respect to the algebraic operations. The purely algebraic way of looking at this business is in terms of **direct limits**.

> A concrete category is said to have **direct limits** if every directed set of objects has a direct limit. If a concrete category has direct limits, then the stalk at any point (in a T_1 space) can be given the structure of an object over the concrete category, namely, as the direct limit of the objects corresponding to open sets containing the point. In particular, all varieties of algebras have direct limits, so the stalks can be given structures of objects.

In an even more abstract vein, a category (not necessarily concrete) that has direct limits has an associated notion of stalk, as the direct limit of the objects corresponding to open sets, about the point. However, because the category is not concrete, the stalk is not naturally a set, and there is no notion of germ!

More generally, for a category with direct limits, given any subset of the topological space, we can take a direct limit of the objects corresponding to open sets containing that subset, and hence associate an object to that set. Note that we are using the T_1 assumption to conclude that every subset is an intersection of the open sets containing it.

New Notation 3. Extending the convention of object associated to open sets, the object associated to an arbitrary subset K of a topological space X by a presheaf $\mathcal{A}$ is denoted as $\mathcal{A}(K)$.

4.3. **Presheaf of sections.** For the presheaf of functions, we had given a topology to the etale space as follows: the basis elements correspond to elements of the form (U, f) where U is an open set and f is a function on it. To each such (U, f), the corresponding basis set is the set of all germs of f at points in U.

The above topology can be generalized to the etale space of a general presheaf (over a concrete category):

Definition. The topology on the etale space is defined by assigning basis sets as follows. Let X be the topological space and $\mathcal{A}$ be the presheaf. For each open set U in X and f in the object corresponding to U, let (U, f) denote the subset of the etale space comprising the germs of f at points in U. Each (U, f) is designated as a basis set.

With this topology, it makes sense to talk of continuous sections of the etale space, just as we did for the presheaf of continuous functions. This gives a new presheaf, called the presheaf of sections:

Definition. The **presheaf of sections**(defined) of a presheaf is the presheaf that associates to each open set in the space, the set of continuous sections from this open set to the etale space. When the category is a variety of algebras, the presheaf of sections can also be given the natural structure of an object in the category by performing operations separately on each stalk.

There is a presheaf homomorphism from the original presheaf to its presheaf of sections, that takes each presheaf element of the form (U, f) where f is an element in the object corresponding to U, to the section that maps each point in U to the germ of f at that point. Further, this map is a presheaf homomorphism in so far as the algebraic structure is concerned.

4.4. **Conjunctive monopresheaf.** While discussing the etale space for the presheaf of functions, we had observed that a continuous section over an open set can be used to provide an open covering of that open set with functions defined on each of the open sets, compatible on intersections. In fact, with identical reasoning, we conclude that much the same happens when we are handling a general presheaf. That is:

> A continuous section over an open set in the space gives rise to an open cover of that open set, with an associated element in the corresponding object for each open set, satisfying compatibility on intersections. That is, a continuous section of U gives an open covering U_w of U and an element x_w in the object corresponding to each U_w, such that the restriction of x_v and of x_w to $U_v \cap U_w$ are the same.

Now, if we were to *start* with an element x in the object corresponding to U, to obtain the continuous section, then each x_w would be the restriction of x to U_w. Given the above data, can we reconstruct the x? And can we do so uniquely? This question motivates two definitions:

(1) A **conjunctive presheaf**(defined) is a presheaf over a concrete category such that if we have a family (U_w, x_w) where U_w are open sets, x_w are elements in the object for U_w, $\bigcup U_w = U$, and the restriction of x_v and x_w to $U_v \cap U_w$ are the same, then there is an element x in the object for U whose restriction to each U_w is x_w.
   Basically, this condition says that if we give the elements locally in a compatible, or consistent fashion, there is an element, globally, that patches them up.
(2) A **monopresheaf**(defined) is a presheaf over a concrete category such that if we have a family (U_w, x_w) where U_w are open sets, x_w are elements in the object for U_w, $\bigcup U_w = U$, and the restriction of x_v and x_w to $U_v \cap U_w$ are the same, then there is at most element x in the object for U whose restriction to each U_w is x_w.
   Basically, this condition says that the patching up is unique.

Clearly, for a conjunctive presheaf, any continuous section over an open set arises via an element in the object corresponding to that open set. Thus, the map from the presheaf to the presheaf of sections is surjective.

Similarly, for a monopresheaf, any continuous section over an open set can arise from at most one element in the object corresponding to that open set. Thus, the map from the presheaf to the presheaf of sections is surjective.

A **conjunctive monopresheaf** is a presheaf that is both conjunctive and a monopresheaf. Thus, a conjunctive monopresheaf is a presheaf such that the map to its presheaf of sections is an isomorphism.

It is clear that the presheaf of sections of any sheaf is both conjunctive and a monopresheaf.

This leads us to a new definition:

Definition. A presheaf is called a **sheaf**(defined) if and only if it is both conjunctive and a monopresheaf. A presheaf is a sheaf if and only if the homomorphism to the presheaf of sections is an isomorphism. The presheaf of sections of any presheaf is a sheaf, and hence, is simply called the "sheaf of sections". The functor taking each presheaf to its sheaf of section is termed the **sheafification functor**(defined).

A little comment here. Every subpresheaf of a monopresheaf is a monopresheaf. The sheaf of all continuous functions is a monopresheaf, and, for this reason, every subpresheaf of this is also a monopresheaf.

On the other hand, a presheaf of functions is *conjunctive* if and only if the corresponding property is *local.*

4.5. **Another definition of sheaf.** Sheaves are often defined in an alternate manner:

Alternative Definition. A **sheaf**(defined) on X with values in a given concrete category, is a topological space (known as the *etale space*) along with a projection map to X that is a local homeomorphism and such that the inverse image of every point (known as the *stalk* at the point) has the structure of an object in the category.

The corresponding presheaf of our old definition is simply the presheaf of sections.

Thus, to define a sheaf, it suffices to give the stalk at each point, and the topology on the etale space.

Moreover, if for a presheaf we give only the stalks at each point, we *lose some information*, and though we can construct the associated sheaf, we cannot find out what the original presheaf was.

4.6. **Inverse image.** We had earlier on defined the **direct image**(recalled) of a presheaf under a continuous map as a presheaf structure that assigns to each open set, the object associated with its inverse image. Via this construction, a presheaf structure on a topological space can give a presheaf structure on another topological space whence there is a continuous function from the first to the second.

The **inverse image**(first used) construction starts with a continuous function and a presheaf structure on the *right* topological space, to obtain a presheaf structure on the left topological space. The idea is as follows:

Definition. Given a continuous function $f : X \to Y$ and a presheaf structure $\mathcal{O}$ on Y, the inverse image of $\mathcal{O}$ via f is the sheaf on X such that the stalk at any point $x \in X$ is the stalk at $f(x)$.

The inverse image of a presheaf is always a sheaf. This is because we are only providing the stalks. Thus, the concept of inverse image makes most sense when we are looking at sheaves only.

New Notation 4. The inverse image of a sheaf $\mathcal{A}$ via f is termed $f^*\mathcal{A}$.

4.7. **Subspace and sheaf structure on subspace.** The *sheaf restriction* starts with a sheaf on a topological space and gives a sheaf structure on a subspace. This makes good sense for sheaves and not for presheaves, because it is a special case of the *inverse image*.

Definition. The **sheaf restriction**(defined) of a sheaf from a space to a subspace is defined as the inverse image of the sheaf under the inclusion map. That is, if $\mathcal{A}$ is a sheaf on a topological space X, and $K \subseteq X$, and the restriction of $\mathcal{A}$ to K is the inverse image of $\mathcal{A}$ under the inclusion map of K in X.

Note that sheaf restriction differs from the restriction map within a sheaf. The restriction map within a sheaf is simply a map between the associated objects to open sets. The sheaf restriction, on the other hand, starts with a sheaf on a space and gives a sheaf on a subspace.

On the other hand, the *extension* map starts with a sheaf structure on the subspace and gives a corresponding sheaf structure on the whole space. It is defined as follows:

Definition. The **extension by zero**(defined) of a sheaf from a locally closed subspace to the whole space is the sheaf whose stalk at each point outside the subspace is zero, and whose stalk at each point inside the subspace is the same as its stalk as defined for the original sheaf.

In symbols, if $K \subseteq X$ and $\mathcal{A}$ is a sheaf on K, then we define the extension of $\mathcal{A}$ by zero to X as having, for each $x \in K$, the same stalk as $\mathcal{A}$ does, and for all other x, the one point stalk $\{0\}$.

New Notation 5. Given a sheaf $\mathcal{A}$ on a subspace K of X, the extension of $\mathcal{A}$ to X is denoted as $\mathcal{A}^X$. Given a sheaf $\mathcal{O}$ on X, the restricted sheaf to a subspace U is denoted $\mathcal{O}|U$ and the extension of this restriction back to X is denoted as $\mathcal{O}_U$. $\mathcal{O}_V$ is naturally a subsheaf of $\mathcal{O}_U$ whenever V is a subspace of U. Note that both are sheaves on the whole space X.

4.8. f **Cohomorphisms.** Given a map $f : X \to Y$ and presheaves $\mathcal{A}$ and $\mathcal{B}$ on X and Y respectively, an f cohomomrphism from $\mathcal{A}$ to $\mathcal{B}$ is a map that, for each open set $U \subseteq Y$, gives a homomorphism from $\mathcal{B}(U)$ to $\mathcal{A}(f^{-1}(U))$ such that the maps commute. Here are some observations:

- The f cohomomorphisms from $\mathcal{A}$ to $\mathcal{B}$ can be naturally identified with the homomorphisms from $\mathcal{B}$ to $f\mathcal{A}$.
- The f cohomomorphisms from $\mathcal{A}$ to $\mathcal{B}$ can be naturally identified with the homomorphisms from $f^*\mathcal{B}$ to $\mathcal{A}$ when they are sheaves.

4.9. **Quotient sheaf.** Suppose X is a topological space with a sheaf $\mathcal{A}$ on it, and $\mathcal{O}$ is a subsheaf of $\mathcal{A}$. Then, for every open set $U \subseteq X$, there is an object $\mathcal{A}(U)$ and an object $\mathcal{O}(U)$. There is also a natural quotient object $\mathcal{A}(U)/\mathcal{O}(U)$. The question is: can we get a sheaf from this?

Indeed we can. This is because given $V \subseteq U$, there is a restriction map from $\mathcal{A}(U)$ to $\mathcal{A}(V)$ and also from $\mathcal{O}(U)$ to $\mathcal{O}(V)$. Consequently, there is a map from $\mathcal{A}(U)/\mathcal{O}(U)$ to $\mathcal{A}(V)/\mathcal{O}(V)$. Thus, we get the *restriction maps* and we can easily see that we have a presheaf. A little more thought reveals that, in fact, it is also a sheaf (in the sense that it is conjunctive and a monopresheaf).

A little definition here (that shall be made clearer in later parts):

Definition. A sequence of sheaves with sheaf homomorphisms is said to be **exact**(defined) if the associated sequence at every point is an exact sequence.

If we have a sequence of sheaf homomorphisms that is exact on every open set, then it is also exact at all points (because the *direct limit* operation preserves exactness). Hence, any sequence of sheaf homomorphisms that is exact on every open set is also exact at every point, and hence, becomes an exact sequence of sheaves. However, the converse is not true: given an exact sequence of sheaves (That is, a sequence of sheaves with homomorphisms that is exact at each point) we may not have exactness at all open sets. The problem is *failure of surjectivity.*

The sequence $0 \to \mathcal{O} \to \mathcal{A} \to \mathcal{A}/\mathcal{O} \to 0$ where $\mathcal{A}/\mathcal{O}$ is the quotient sheaf is exact at all open sets, not just at points.

A general question that we shall discuss in later parts is: "under what conditions is exactness at points good enough for exactness on open sets?"

5. Property theory of sheaves and presheaves

5.1. **Metaproperties and property operators.** Here are some things we can do with sheaves and presheaves:

- Pass to the associated sheaf
- Take direct image under a continuous map
- Take inverse image under a continuous map
- Pass to a subspace,or pass to a subspace and extend back
- Take a subpresheaf or a subsheaf

This results in the following metaproperties of sheaf and presheaf properties:

- A presheaf property is said to be **sheaf inherited**(defined) if the associated sheaf for a presheaf having the property also has the property.
- A presheaf property is said to be **conjunctive to sheaf inherited**(defined) if the associated sheaf for a conjunctive presheaf having the property also has the property.
- A presheaf property is said to be **mono to sheaf inherited**(defined) if the associated sheaf for a monopresheaf having the property also has the property.
- A presheaf property is said to be **direct image preserved**(defined) if the direct image of a presheaf having the property also has the property.
- A presheaf property is said to be **inverse image preserved**(defined) if the inverse image of a presheaf having the property also has the property. Any inverse image preserved presheaf property must be sheaf inherited.
- A presheaf property is said to be **subpresheaf hereditary**(defined) if any subpresheaf of a presheaf having the property also has the property.
- A presheaf property is said to be **subspace hereditary**(defined) if given a presheaf having the property, the associated presheaf on a subspace also has the property.
- A presheaf property is said to be **extension preserved**(defined) if any extension of a presheaf having the property on a subspace, to the whole space, also has the property there..
- A presheaf property is said to be **subspace extension hereditary**(defined) if given a presheaf having the property, the extension to the whole space of the presheaf associated with a subspace also has the property.

5.2. **Flabbiness and softness.** One area of study is the nature of the restriction maps. A presheaf is said to be **flabby**(defined) if the restriction map from the whole space to any open set is surjective. Alternatively, any element in an object corresponding to an open set can be extended to an element in the object corresponding to the whole space.

Flabbiness implies, in fact, that all the restriction maps are surjective. In other words, elements over smaller open sets can always be lifted to elements over bigger open sets.

A **soft subset**(defined) of a topological space with respect to a presheaf is a subset such that given any element over any open set containing that subset, there is an element over the whole space and a smaller open set containing that subset such that the restriction of the global element and the original element to this open subset are the same. In other words, a soft subset is a subset such that every element in the associated object obtained by taking direct limits on it arises from a global element.

Given a family Φ of subsets, the presheaf is Φ **soft**(defined) if every member of Φ is soft. There are three important cases for Φ:

- When Φ is the family of open subsets, Φ can also be taken as the family of all subsets, and the presheaf is flabby.
- When Φ is the family c of compact subsets, the presheaf is called c **soft**(defined)
- When Φ is the family *cld* of closed subsets, the presheaf is called **soft**(defined)

Here, now, are some observations:

- Flabbiness is conjunctive to sheaf inherited.
- Softness, over a paracompactifying family of supports, is conjunctive to sheaf inherited.

These points shall not concern us in this part. They will be picked up again later.

Visual Analogy 5. *Softness and flabbiness have rather precise meanings in sheaf theory, as opposed to their loose meanings in the English language. Flabby sheaves can be thought of as being* very roomy*, in the sense that there are* lots of global elements*. In particular, for a flabby sheaf, any element over an open set can be extended to the whole space.*

Soft sheaves are also quite roomy, though not as much as flabby sheaves. Being able to extend only from closed sets puts certain sanity restrictions on what elements can be extended. For instance, we will not be able to extend continuous functions on an open set that do not take definite limiting values at points on the closure.

6. Topological properties of the etale space

6.1. **Sheaf properties on the etale space.**

A sheaf is given by describing an etale space and a projection map from the etale space to the original space. Thus, topological properties of the etale space give rise to properties of the sheaf. Here, we discuss the typical classes of topological properties for the etale space and what they mean for the sheaf.

6.1.1. *Separation axioms.*

The main idea in **pointwise separation axioms** is that if the starting topological space satisfies a separation axiom, then the etale space will satisfy the separation axiom when the points f the etale space have distinct projections. The problem largely arises when they have the same projection.

Consider the T_1 axiom. Suppose that the starting space is T_1. Then, given two points in the etale space who images on the space are distinct, all we need to do is:

- Find an open set in the original space containing the one point and not the other.
- Now, look at the germ. Consider an element in an open set contained inside this open set, whose germ is the given germ.

Thus, two points whose images differ are separated. What happens if the two points correspond to the same point in the starting space but different germs? Then, any element on an open set whose germ is the one gives an open set containing the one and not the other.

Thus, if the starting space is T_1, so is the etale space.

For Hausdorffness, the analogue of the above argument goes through if the starting points are distinct. But if it is the same point and the germs are distinct, then the argument does not go through. What we manage to conclude is:

> The etale space for a presheaf over a Hausdorff space is again Hausdorff if and only if the following holds: if two functions have different germs at a point, then there is an open neighbourhood about that point such that the two functions have different germs at *each* point in that open neighbourhood.

The above condition is fairly demanding, and therefore, in most circumstances the etale space is far from Hausdorff.

Points to Ponder

- What are the necessary and sufficient conditions for the etale space to be regular, completely regular, normal?
- Is the etale space always locally Hausdorff? What are the conditions to ensure it is locally regular, locally completely regular, and locally normal?

6.1.2. *Connectedness.* When is the etale space connected? When is it path connected? Is it locally connected?

Here are some observations to begin with:

- The induced topology on each stalk is discrete.
- If the original space is connected, then for any partition of the etale space into nonempty disjoint open sets, both the open sets must map surjectively to the original space.
- Clearly, every basis set corresponding to a connected open set in the original topological space, is connected. Therefore, for any partition of the etale space into nonempty disjoint open sets, it must lie in one of the open sets.
- Thus, any element over a connected open set in the original topological space must have all its germs in one open set or the other.

In essence, a partition into disjoint open sets means a partition of all the elements over connected open sets into two classes, such that no two elements over different classes have the same germ at *any* point. Another way of putting this is:

> A partition of the etale space into disjoint open sets corresponds to a property for elements over connected open sets, such that the property is local and its complement is also local.

This leads us to the following:

> If, given any two germs at two distinct points, there is an element over a connected open set having those two germs, then the etale space is connected.

In particular, we can readily verify that:

- The etale space for a flabby sheaf over a connected Hausdorff space is connected. Indeed, all we need to do is: get elements over disjoint connected open sets containing both the points, and then, by flabbiness, simply extend to an element of the whole space.
- The etale space for a soft sheaf over a connected regular space is connected. All we do is take a shrinking of the original open set.

Points to Ponder

- Are the analogous statements true when "connected" is replaced by "path connected"?
- Under what conditions is the etale space locally connected?

6.1.3. *Compactness.* We begin by observing that every stalk in the etale space is a closed discrete subset. From this we conclude:

- The etale space is compact only if the original space is compact and the stalks are all finite – something which does not happen in the cases of interest.

6.2. **The projection map.** Here are some definitions:

- A **fiber map** is a map of topological spaces $\pi : E \to X$ with a fixed space F such that for each $x \in X$, there is an open set U about x so that $\pi^{-1}(U)$ is isomorphic to $U \times F$ with π the projection map.
- A **covering map** is a fiber map where F is a discrete space.
- An **open map** is a map that takes open sets to open sets.
- A **closed map** is a map that takes closed sets to closed sets.
- A **quotient map** is a surjective map that takes saturated open sets to open sets.
- A **local homeomorphism** is a continuous map such that for each point in the domain, there is an open neighbourhood about it such that the restriction to that neighbourhood is a homeomorphism.

Every covering map is a fiber map, and every fiber map is an open map as well as a local homeomorphism. Covering maps need not, however, be closed in general.

If the stalks at all points are isomorphic (which is not an unreasonable assumption) then the projection map from the etale space is a covering map. In particular, it is an open map and a local homeomorphism.

Points to Ponder

- Under what conditions is the projection map from the etale space to the underlying topological space closed?

7. Some kinds of functions

7.1. **Continuous functions.** If we associate, to a topological space X, the presheaf of continuous functions from open sets in X to a topological space Y, this presheaf naturally becomes a sheaf.

Continuous functions to $\mathbb{R}$ or $\mathbb{C}$ form a **soft sheaf** when X is normal. That is, any continuous function from a closed set to $\mathbb{R}$ can be extended to a continuous function from the whole space to $\mathbb{R}$. Actually, the space $\mathbb{R}$ or $\mathbb{C}$ may be replaced by any sheaf that has the so called **Universal Extension Property**.

(Unless stated otherwise, we assume $Y = \mathbb{R}$ or $Y = \mathbb{C}$).

The etale space for the sheaf of continuous functions is **connected** whenever X is a connected normal space and Y has the Universal Extension Property. This is because we get a soft sheaf on a connected regular space.

The etale space of continuous functions need not be Hausdorff, in fact, usually, it is not. This is because there may be functions that have different germs at a point but have the same germ on a sequence of points converging to that point.

7.2. **Constant functions and polynomial functions.** The property of a function being a constant function is not local, so the constant functions do not form a sheaf. However, the **locally constant** functions form a sheaf. The presheaf is connected-conjunctive, but not conjunctive. The presheaf of constant functions is soft, even flabby, but the sheaf of locally constant functions is not soft. In fact, the etale space is not even connected.

Polynomial functions on an affine space again form a presheaf, which is conjunctive for connected unions of open sets, but not conjunctive in general. The presheaf is again flabby, even soft, but the associated sheaf, viz the sheaf of **locally polynomial functions**, is neither. Again, the etale space of the sheaf of locally polynomial functions is not even connected.

7.3. **Hard presheaves.**

Definition. Here are some definitions I have introduced:

- A subset of a presheaf is called **hard**(defined) if the germ of an element defined in the corresponding object is uniquely determined by its germ at any point. Thus, $K \subseteq X$ is hard for a presheaf $\mathcal{A}$ if $\mathcal{A}(K)$ is uniquely determined by its restriction to any point.

  The restriction map defined from a hard open subset to any smaller open subset is injective.
- A presheaf is termed **hard**(defined) if every open subset (and hence every subset) is hard.
- A presheaf is termed **locally hard**(defined) if every point has a hard open neighbourhood.

The presheaf of constant functions is both hard and flabby. In other words, the restriction maps are both injective and surjective (and hence they are all isomorphisms). Similarly the presheaf of polynomial functions on $\mathbb{R}$ is both hard and flabby.

The question that we would like to answer is: what happens when we apply the sheafification functor to a hard, flabby and connected-conjunctive sheaf?

- The associated sheaf to a hard presheaf that is conjunctive for connected coverings has the property that every connected open subset is hard. In particular, it is locally hard.

  In other words, for connected open subsets, it remains true that the function on the whole subset is determined uniquely by specifying the germ at any point.
- The associated sheaf to a hard presheaf cannot be flabby or even soft, unless all the stalks are trivial. This can be shown as follows: take any two points in the space, and consider incompatible germs on them. Then clearly, these cannot be extended to a function on the whole space.
- The associated sheaf to a hard presheaf is not connected, because we can very easily separate functions by their germs.
- The associated etale space to a hard, or even locally hard presheaf, is Hausdorff. In fact, any point that has arbitrarily small hard connected open sets containing it allows separation of germs over it.

Concept Testers

(1) Under what conditions is the etale space associated with a given hard presheaf normal?

7.4. **Differentiable functions.** In this part, we only talk of differentiable functions from open sets in $\mathbb{R}^n$ to $\mathbb{R}$. In the next part, we shall extend the notion to differentiable functions on arbitrary differential manifolds.

Given a function f from an open set $U \subseteq \mathbb{R}^m$ to $\mathbb{R}^n$, the **total derivative**(defined) of f at a point $x \in U$ is the $n \times m$ matrix T such that such that as $h \to 0$ the expression:

$$\frac{||f(x+h) - f(x) - Th||}{||h||}$$

tends to 0. In other words, $f(x) + Th$ is a linear approximation to $f(x+h)$.

Thus, the total derivative of the function is from the open set to $\mathbb{R}^{mn}$.

Here are some basic facts about differentiation of functions $\mathbb{R}^m$ to $\mathbb{R}^n$:

- The function has a total derivative if and only if composing it on the left with the coordinate projections in $\mathbb{R}^n$ gives a function with a total derivative. Moreover, the components of the total derivative are the total derivatives of these composites. Thus, it suffices to consider functions from $\mathbb{R}^m$ to $\mathbb{R}$.
- If the function has a total derivative at a point, then it has partial derivatives with respect to each of the coordinates, and each partial derivative is a component of the total derivative.
- If the partials exist at the point, the function may not have a total derivative at the point. However, if the partials exist and are continuous, then the function has a continuous total derivative at the point. Conversely, if the total derivative is continuous, then the partials exist and are continuous.

When we use the term *function* we shall mean functions from open sets in $\mathbb{R}^m$ to $\mathbb{R}^n$.

A function is said to be **differentiable**(defined) if it has a total derivative at every point. The k^{th} **derivative**(defined) is obtained by computing the total derivative k times. A function is said to be **continuously differentiable**(defined) if it has a total derivative at every point and the total derivative mapping is continuous. A function is said to be C^k(defined) if it is k times differentiable and the k^{th} derivative is continuous.

A function is said to be C^∞(defined) if it is C^k for every k..

In global calculus, when we talk of differentiable functions, we usually mean C^∞ functions to $\mathbb{R}$.

Now we come to the big result:

Claim. If U is a shrinking of V (both are open sets), then there is a differentiable function that is 1 on the whole of U, and 0 outside V.

The typical function used for this is the composite of $x \mapsto 1/x$ with a Schwartz function.

Note that this result is equivalent to the statement:

Claim. Let X be an open subspace of $\mathbb{R}^m$. Then, any two closed subsets of X are separated by a differentiable function to $[0, 1]$ taking one closed set to 0 and the other to 1.

This is a kind of extension of the analogous property for *continuous functions* on normal topological spaces.

7.5. **Locally analytic functions.** A **locally analytic function** on an open set in $\mathbb{R}^n$ is a function that, for each point in the open set, can be expanded locally about that point in a power series in terms of the coordinates with respect to that point. Clearly, locally analytic functions form a sheaf. We shall look at this from the viewpoint of hard presheaves in a later part.

8. Sheaves of rings

8.1. **A quick explanation.** The two concrete categories in which we shall consider sheaves to take values are:

- The category of (unital, commutative) rings, or $\mathbb{R}$ or $\mathbb{C}$ algebras
- The category of modules over a given ring, or bimodules over a given pair of rings

We've already seen some examples of both from the viewpoint of global calculus. The algebraic geometry viewpoint gives different sheaves where we still work over the category of rings. In this section, we introduce some terminology that is useful both for global calculus and for algebraic geometry, though its primary use is in the latter.

8.2. **Ringed space.**

Definition. A **ringed space**(defined) is a topological space along with a sheaf of commutative unital rings. The associated sheaf is termed the **structure sheaf**(defined).

Because this definition is primarily motivated by algebraic geometry, it is often not even true that the space in question is Hausdorff, though it usually is T_1. A somewhat more special case that is usually of interest is:

Definition. A **locally ringed space**(defined) is a ringed space such that the stalk at every point is a local ring.

A k locally ringed space is a locally ringed space where all the quotients are isomorphic to k.

8.3. **Moving from rings to ringed spaces.** Given a ring R, and an ideal I, we can consider the quotient ring of R by I. Also, given an R module M, the quotient of M by the submodule IM is a module over this quotient ring.

In the particular case where R is a local ring and I is its unique maximal ideal, the quotient ring is a field. Hence, for any R module M, the quotient of M by IM is a vector space over this quotient field.

Generalizing to sheaves, we get:

- Given a ringed space and a subsheaf of the structure sheaf whose stalk at each point is an ideal of the ring stalk, there is a natural quotient sheaf, whose stalk at each point is the quotient ring. Moreover, for any module of sheaves over the ringed space, the quotient by the submodule of sheaves whose stalks are products of its stalks with the ideal, is a module over the quotient sheaf.
- Given a k locally ringed space, the quotient sheaf by the maximal ideal subsheaf has stalks at each point as k vector spaces. Moreover, for any module of sheaves over this k locally ringed space, the quotient by the submodule of sheaves whose stalks at each point are the products of its stalks with the maximal ideal, is a module over the quotient sheaf. Because the quotient sheaf has a copy of k at each point, this is called a k vector bundle. We shall see more on vector bundles later.

8.4. **Tangent space.** Here, I give a couple of definitions. These shall be elaborated upon later:

- The **cotangent space**(defined) or **cotangent bundle**(defined) is the space whose stalk at each point x is m_x/m_x^2. The cotangent space of a k locally ringed space gives, at each point, a k vector space.
- The **tangent space**(defined) or **tangent bundle**(defined) is the dual of the cotangent space.

The idea is that each element in the tangent space must give a rule for differentiating elements of R_x. As m_x has a complement, namely, the ring of constant functions, describing the differentiation rule over m_x is sufficient. Further, we know that within m_x, by the so called "Leibniz rule", all elements in m_x^2 map to 0, and so, by linearity, it is enough to give the function on the quotient. Thus, the tangent space comprises functionals on m_x/m_x^2.

9. What comes next

9.1. **The definition of differential manifold.** In the first few sections, we gave a good starting idea of how a differential manifold is to be defined. At that time, however, we did not have the proper machinery of terminology and notation to given a formal definition. The notions of presheaf and sheaf have been introduced in the later sections to pave the way for the formal definition.

In the next article, we shall define a differential manifold, as well as some related ideas, formally. We shall try to figure out the sheaf theoretic properties and phenomena in the context of differential manifolds. We shall also take a quick look at topological manifolds, real analytic manifolds, and complex manifolds. We will also discuss various phenomena observed for $\mathbb{R}^n$ and open sets in $\mathbb{R}^n$, in the general context of differential manifolds. Finally, we will take a quick look at the algebra of differential operators.

9.2. **More on sheaves.** In the third article, we will take up the theory of sheaves a little further, and try to understand the *various scenarios in which sheaves and presheaves naturally turn up* and what properties of them we might expect based on the scenario. The study will be largely property theoretic.

We will also explore the concept of **sheaf theoretic cohomology**. This shall pave the way for the cohomology theory of differential manifolds.

9.3. **Back to differential manifolds.** In the fourth article, we will return to a study of differential manifolds, and view their cohomology. Another direction of exploration that will surface is the study of Lie groups and Lie algebras. We will be able to view differential operators from a new perspective.

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