

# WHITNEY EMBEDDING THEOREM

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ABSTRACT. The Whitney embedding theorem states that any smooth manifold of dimension  $n$  can be embedded as a closed submanifold of  $\mathbb{R}^{2n}$ . In this article, I prove a weaker version of the theorem, which states that any smooth manifold of dimension  $n$  can be embedded as a closed submanifold of  $\mathbb{R}^{2n+1}$ . I do this in two steps: first, embed it in  $\mathbb{R}^N$  for some finite  $N$ , and then, through iterated projections, push the dimension to  $2n + 1$ .

## 1. THE WEAK FORM

### 1.1. Statement.

**Theorem 1** (Whitney embedding: weak form). A smooth compact real manifold of real dimension  $n$  can be embedded as a closed submanifold of  $\mathbb{R}^N$  for some  $N \geq n$ .

Some remarks:

- The *closed* part arises automatically because any compact subset of a Hausdorff space is closed.
- It suffices to restrict our attention to connected components. This is because every compact space has only finitely many connected components, and if we obtain embeddings of each, then we can obtain embeddings of their union by simply placing the images sufficiently far away.

### 1.2. Definition of manifold.

**Definition.** A **manifold**<sub>(defined)</sub> of dimension  $n$  is a topological space  $S$  along with an open cover  $U_i$ ,  $i$  ranging over an index set  $I$ , and homeomorphisms  $\phi_i : U_i \rightarrow V_i$  with  $V_i$  open in  $\mathbb{R}^n$ .

- Each open set  $U_i$  along with the mapping  $\phi_i$  is termed a **coordinate chart**<sub>(defined)</sub>.
- The collection of coordinate charts corresponding to an open cover is termed an **atlas**<sub>(defined)</sub>.

### 1.3. Definition of differential manifold.

**Definition.** A **differential manifold**<sub>(defined)</sub> of dimension  $n$  is a topological space  $S$  along with an open cover  $U_i$ ,  $i$  ranging over an index set  $I$ , and homeomorphisms  $\phi_i : U_i \rightarrow V_i$ , where  $V_i$  is open in  $\mathbb{R}^n$ , such that the following holds:

Consider the set  $U_i \cap U_j$ . Let  $W$  denote  $\phi_i(V_i \cap V_j)$ . Then the composite mapping  $\phi_j \circ \phi_i^{-1}$  is a diffeomorphism between open sets in  $\mathbb{R}^n$ .

### 1.4. Concept of differentiable embedding.

**Definition.** Let  $M_1$  and  $M_2$  be two differential manifolds. A **differentiable embedding**<sub>(defined)</sub> of  $M_1$  in  $M_2$  is a mapping  $f : M_1 \rightarrow M_2$  such that:

- $f$  is injective, viz  $f(p) = f(q) \implies p = q$ .
- At any point  $p \in M_1$ , the mapping  $df_p : T_p(M_1) \rightarrow T_{f(p)}M_2$  is injective. That is, if  $v$  and  $w$  are two tangent vectors at  $p$ , then  $df_p(v) = df_p(w) \implies v = w$ .

With this definition, we are in a position to understand the statement of the Whitney embedding theorem: given any differential manifold that is *compact*, *connected* of dimension  $n$  as a topological space, there is a differentiable embedding into  $\mathbb{R}^N$ .

### 1.5. Compactness translates to a finite atlas.

**Observation 1.** A compact manifold has a finite atlas, that is, there is a finite collection of coordinate charts  $U_i$  with mappings  $\phi_i$  to open sets  $V_i$  in  $\mathbb{R}^n$ .

*Proof.* Take any atlas of the manifold. Choose a finite subcover of the open cover for this atlas, and we have a finite atlas.  $\square$

### 1.6. Partition of unity.

**Definition.** Let  $X$  be a topological space, and  $U_i$  be a point-finite open cover of  $X$ . A **partition of unity**<sub>(defined)</sub> corresponding to  $U_i$  is a collection of continuous functions  $f_i : X \rightarrow [0, 1]$  such that the support of  $f_i$  is contained in  $U_i$  for each  $i$ , and such that at each point  $p$ , the sum of the values  $f_i(p)$  is 1.

**Theorem 2.** Given a compact space and a finite open cover of it, there exists a partition of unity for that finite open cover.

### 1.7. Cartesian product of all the functions.

**Definition.** Let  $f_i : A \rightarrow B$  be a collection of  $n$  functions. Then the Cartesian product of  $f_i$  is defined as a function  $f : A \rightarrow B^n$  such that the  $i^{\text{th}}$  coordinate projection of  $f$  is  $f_i$ .

We now have all the ingredients for the proof.

**1.8. The embedding.** Let  $M$  be a compact  $n$ -dimensional differential manifold, and  $(U_i, \phi_i)$  be a finite atlas (with  $m$  elements). Let  $f_i$  be a partition of unity corresponding to the open cover  $U_i$ . Consider a mapping  $\mu_{ij} : p \mapsto x_i(\phi_j(p))f_j(p)$  where  $x_i$  denotes the  $i^{\text{th}}$  coordinate. Clearly,  $\mu_{ij}$  are zero outside  $U_i$ , because the support lies there, and they are also continuous inside  $U_i$ , because there they are the products of two continuous functions. Take the direct product of all the  $\mu_{ij}$ s and all the  $f_j$ s. Call this direct product  $\Phi$ .  $\Phi$  is a mapping from  $M$  to  $\mathbb{R}^{m(n+1)}$ .

**Claim.** The mapping  $\Phi$  described above is injective.

*Proof.* Suppose  $p$  and  $q$  are two distinct points in the manifold  $M$  such that  $\Phi(p) = \Phi(q)$ . Then clearly, there is some  $f_j$  such that  $f_j(p) = f_j(q) \neq 0$  (the not equal to zero comes because the  $f_j$ s form a partition of unity). Thus, both  $p$  and  $q$  lie inside the ball  $U_j$ .

Note that  $f_j(p) = f_j(q) \neq 0$ , and the local coordinate chart  $\phi_j$  must have at least one differing coordinate for  $p$  and  $q$ . If this is the  $i^{\text{th}}$  coordinate, then  $x_i(f_j(p)) \neq x_i(f_j(q))$ , and hence  $\Phi(p) \neq \Phi(q)$ , leading to a contradiction.  $\square$

**Claim.** The differential of  $\Phi$  is injective.

*Proof.* Because writing every coordinate of the differential is cumbersome, I abbreviate the expression by clubbing together the coordinates for the differentials in the directions  $\mu_{ij}$  with the same  $i$  and varying  $j$ . That is, I club these together as vectors.

Then, applying the product rule, the differential of  $\Phi$  at  $p$  sends an element  $v \in T_p M$  to:

$$((df_1)_p(v)\phi_1(p) + f_1(p)(d\phi_1)_p(v), \dots, (df_k)_p(v)\phi_k(p) + f_k(p)(d\phi_k)_p(v), (df_1)_p(v), \dots, (df_k)_p(v))$$

where  $(df_i)_p(v)\phi_i(p) + f_i(p)(d\phi_i)_p(v)$  is itself a  $n$ -tuple.

Suppose, at a point  $p$ , there are two vectors  $v$  and  $w$  such that  $d\Phi_p(v) = d\Phi_p(w)$ . Then,  $(df_k)_p(v) = (df_k)_p(w)$  for all functions  $f_k$ . Because the  $f_i$ s form a partition of unity, there is some  $f_i$  such that  $f_i(p) \neq 0$ . Putting  $k = i$ , we obtain that  $(df_i)_p(v) = (df_i)_p(w)$ .

Also, the vectors  $(df_i)_p(v)\phi_i(p) + f_i(p)(d\phi_i)_p(v)$  and  $(df_i)_p(w)\phi_i(p) + f_i(p)(d\phi_i)_p(w)$  are equal. This gives us:

$$\begin{aligned} (df_i)_p(v)\phi_i(p) + f_i(p)(d\phi_i)_p(v) &= (df_i)_p(w)\phi_i(p) + f_i(p)(d\phi_i)_p(w) \\ \implies f_i(p)(d\phi_i)_p(v) &= f_i(p)(d\phi_i)_p(w) \text{ because } (df_i)_p(v) = (df_i)_p(w) \\ \implies (d\phi_i)_p(v) &= (d\phi_i)_p(w) \text{ because } f_i(p) \neq 0 \end{aligned}$$

$\implies v = w$  because each  $\phi_i$  is a differentiable embedding

Hence, starting that  $d\Phi_p(v) = d\Phi_p(w)$  we obtain  $v = w$ . Thus, the mapping  $d\Phi$  is injective on every tangent space.  $\square$

## 1.9. The “put-together proof”.

## 2. THE FORM WE WANT

**Theorem 3** (Whitney embedding: medium form). Any smooth compact real manifold of real dimension  $n$  can be embedded as a closed submanifold of  $\mathbb{R}^{2n+1}$ .

The proof idea is to show that if there is an embedding in  $\mathbb{R}^N$ , then there is also an embedding in  $\mathbb{R}^{N-1}$  provided  $N > 2n + 1$ .

Roughly speaking, we start with an embedding in  $\mathbb{R}^N$ , and then try to locate a hyperplane such that projecting the embedding on that hyperplane gives a differentiable embedding. In order to do this, we need a result that uses the fact that the *actual dimension of the manifold is low* to argue that *there is a direction* such that all lines in that direction intersect the manifold exactly once, and hence projecting to the perpendicular hyperplane maintains injectivity. We also need to show that since *the actual dimension of the manifold is low*, the differential mapping is also injective.

The result we use for this purpose is Sard’s theorem, or rather a corollary of Sard’s theorem.

**2.1. Sard’s theorem: a recall for application.** First, some terminology:

**Definition.** Let  $f : M_1 \rightarrow M_2$  be a differentiable mapping between differential manifolds  $M_1$  and  $M_2$ .

- (1) A **regular point**<sub>(defined)</sub> for  $f$  is a point  $p \in M$  such that the induced mapping  $df : T_p M_1 \rightarrow T_{f(p)} M_2$  is surjective.
- (2) A **critical point**<sub>(defined)</sub> for  $f$  is a point that is not a regular point.
- (3) A **critical value**<sub>(defined)</sub> for  $f$  is a point in  $M_2$  that occurs as the image of a critical point via  $f$ .
- (4) A **regular value**<sub>(defined)</sub> for  $f$  is a point in  $M_2$  whose inverse image is nonempty and does not contain any critical points.

**Theorem 4** (Sard’s theorem). Let  $f : M_1 \rightarrow M_2$  be a differentiable map between differential manifolds  $M_1$  and  $M_2$ . Then, the set of critical values of  $f$  has Lebesgue measure zero in  $M_2$ .

An immediate consequence of this is what we’ll be using:

**Corollary 1** (Mapping lower to higher dimensions). Suppose  $M_1$  and  $M_2$  are differential manifolds, such that the dimension of  $M_1$  is strictly smaller than the dimension of  $M_2$ . Then, the image of  $M_1$  in  $M_2$  is a set of measure zero.

*Proof.* Because the dimension of  $M_1$  is strictly smaller than the dimension of  $M_2$ , the map  $df$  can never be surjective. Hence, every point is a critical point, and thus, every point in the image is a critical value. Thus, applying Sard's theorem (theorem 4) we obtain that the image has measure zero.  $\square$

## 2.2. Assuring injectivity.

**Lemma 1** (Full-measure set of hyperplanes). Suppose  $N > 2n + 1$ . Suppose there is an embedding  $\Phi$  of an  $n$ -dimensional differential manifold  $M$  in  $\mathbb{R}^N$ . Then, there exists a full-measure set of hyperplanes  $\Pi$  in  $\mathbb{R}^N$  such that the composite mapping  $p_\Pi \cdot \Phi$  is injective from  $M$  to  $\Pi$ .

*Proof.* Note that  $p_\Pi \cdot \Phi$  is injective if and only if every line orthogonal to  $\Pi$  intersects the manifold exactly one point. In other words, we want that *most* directions are *not* available as directions of lines joining pairs of points in the manifold. This reduces to demanding that the mapping from pairs of distinct points in the manifold, to *directions* in  $\mathbb{R}^N$ , have an image of measure zero.

Let's put this formally.

Let  $\Delta$  be the diagonal in  $M \times M$ . Define a mapping:

$$a : M \times M \setminus \Delta \rightarrow \mathbb{R}\mathbb{P}^{N-1}$$

where  $a(x, y)$  is the line through the origin parallel to the line joining  $\Phi(x)$  and  $\Phi(y)$  in  $\mathbb{R}^N$ .

Note that the left hand side has dimension  $2n$ , while the right hand side has dimension  $N - 1$ , which is at least  $2n + 1$ . We can apply corollary 1 to the map  $a$  and conclude that the image of  $a$  is a measure zero set. Hence, the complement is a full-measure set, and thus, the collection of hyperplanes  $\Pi$  for which  $p_\Pi \cdot \Phi$  is injective is a full-measure set.  $\square$

## 2.3. Assuring a differentiable embedding.

**Lemma 2** (Differential injectivity). Suppose  $N > 2n + 1$ . Suppose there is an embedding  $\Phi$  of an  $n$ -dimensional differential manifold  $M$  in  $\mathbb{R}^N$ . Then, there exists a full-measure set of hyperplanes  $\Pi$  in  $\mathbb{R}^N$  such that the composite mapping  $p_\Pi \cdot \Phi$  has injective differential everywhere.

*Proof.* Injectivity of the differential translates to the condition that the line orthogonal to the hyperplane does *not* occur as the image of any tangent space via the differential mapping. Thus, we need to make sure that the differential mapping misses out a full-measure set of vectors.

The differential mapping is a mapping from *tangent directions* to the manifold to *directions* in  $\mathbb{R}^N$ . Let's formalize this.

Consider the projectivization of the tangent bundle of  $M$ , denote as  $\mathbb{P}(TM)$ . This is a fiber bundle over  $M$  with fiber  $\mathbb{R}\mathbb{P}^{n-1}$ . The total space is thus a smooth manifold of dimension  $2n - 1$ . Define a mapping:

$$b : \mathbb{P}(TM) \rightarrow \mathbb{R}\mathbb{P}^{N-1}$$

given as  $l \in T_x(M) \mapsto D_x(\Phi(l)) \in \mathbb{R}^N$ .

The dimension of the left side is  $2n - 1$ , while the dimension of the right side is at least  $2n + 1$ . Hence, we can apply corollary 1 and conclude that the image has measure zero. Hence, the collection of hyperplanes  $\Pi$  for which the differential is injective, is of full measure.  $\square$

## 2.4. Putting the pieces together.

We have obtained that:

- For a full-measure set of hyperplanes, projecting the embedding to the hyperplane gives an injective mapping.

- For a full-measure set of hyperplanes, projecting the embedding to the hyperplane gives a mapping with differential injective.

Intersecting these two full-measure sets, we obtain another full-measure set of hyperplanes where both  $p_{\Pi} \cdot \Phi$  and the differential are injective.

Hence, starting with a differentiable embedding of  $M$  in  $\mathbb{R}^N$ , we have obtained a differentiable embedding of  $M$  in  $\mathbb{R}^{N-1}$ .