Prerequisites: Sufficient for a broad understanding of the talk.

- **Commutative algebra**: Module, ring, algebra, ideal

- **Group theory**: Group action, fixed point, isotropy subgroup, orbit

- **Topology**: Open, closed, dense, closure

- **Generators and relations**: Generating set, relations among generators, free objects, linear dependence

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*Expected time: 2 minutes

Nice to know:

- **More ring theory**: Tensor product, prime ideal, radical ideal

- **Algebraic geometry**: Zariski topology and Hilbert's nullstellensatz

- **Lie theory**: Lie group, Lie algebra, universal algebra, differential operator

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*Expected time: 0 minutes

Module theoretic freeness over invariant subring

Vipul Naik (under Prof Dipendra Prasad)

July 13, 2006

Goals of the talk: I want to

- Give sufficient conditions for Kostant's first and (if time permits) second problem

- Get and give a flavour of invariant theory

- Gain experience in presenting material

- Show how commutative algebra, algebraic geometry, and Lie theory merges

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*Expected time: 0 minutes

Setup: R is a commutative unit ring

- R algebra: Ring containing R as a subring

- R module: Abelian group with R action

R algebra $\implies$ R module

Questions: S is an R algebra. Hence S is an R module. Is S free as an R module?

Remark: A *module theoretic* generating set is very different from an *algebra theoretic* generating set.

[*]

[*]Expected time: 3 minutes

Setup:

- k is a field.
- S is a k algebra.
- G acts as algebra automorphisms on S.
- $J = S^G$ is the invariant subring.

Questions: S is a J algebra. Is S free as a J module?

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*Expected time: 2 minutes

Setup:

- X a set, k a field
- $S \leq k^X$
- G acts on X, inducing algebra automorphisms on S
- $J = S^G$

Questions: Is S free as a J module?

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*Expected time: 0 minutes

Exploration: A function is G invariant iff it is constant on every orbit under G action.

- I_O is the ideal of functions vanishing on orbit O.

- $R_O = I_O + k$ is the ring of functions constant on O.

- $$\bigcap_O R_O = J$$

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*Expected time: 2 minutes

†Blackboard: Orbit picture and explanation

Setup:

- X is a set, k a field
- $S = k[X]$ is the polynomial ring
- $G \leq GL(X)$ and acts on S as degree preserving automorphisms.

Questions: Is S a free module over J? This is **Kostant's First Problem** and I will provide sufficient conditions for a *yes*.

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*Expected time: 1 minute

Setup: $S(O_x)$ is the ring of functions on O_x as restrictions of elements of S.

Exploration: *Free* means absence of non-trivial linear relations. We relate linear independence in two ways:

- Over J versus over k: Note that all k modules are free as k is a field.

- On the whole space (that is, in S) versus restricted to an orbit (that is, in $S(O_x)$)

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*Expected time: 1 minute

†Blackboard: orbit picture, text about restriction

Setup:

- $X = k^n$, $S = k[X]$, $G \leq GL(X)$, $J = S^G$

- $J^+ \equiv$ positive degree part of J

- H is a graded subspace complement to J^+S

Basic result:

$$JH = S$$

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*Expected time: 1 minute

Result (stated without proof): The following are equivalent.

1. $J \otimes H \to S$ (given as $f \otimes g \mapsto fg$) is an isomorphism.

2. S is free over J.

3. Let M be a k submodule intersecting J^+S trivially. Then for elements of M, k linear independence equals J linear independence.

Goal: To determine when equivalent condition (2) holds.

Approach: Try to determine when equivalent condition (3) holds.

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*Expected time: 2 minutes

†Blackboard: Explanation, answering questions

Programme: We prove equivalent condition (3): If M intersects J^+S trivially, nontrivial J linear relation gives nontrivial k linear relation. We take the *orbit* route

- nontrivial J linear relation $\Longrightarrow$ nontrivial k linear relation restricted to orbit

- nontrivial k linear relation restricted to orbit $\Longrightarrow$ nontrivial k linear relation

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*Blackboard: Scribble picture for idea

Claim: nontrivial J linear relation over S $\Longrightarrow$ nontrivial k relation on restriction to orbit

Naive Proof Idea: Map original relation via quotient map to a k linear relation.

Bug: Nontrivial relation may restrict to trivial relation.

Proper proof: Observe that the set of points at which the relation restricts to a trivial one is open. How? Look at the next slide.

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*Expected time: 2 minutes

†Blackboard: Scribble picture to explain difficulty

Claim: Take a collection f_i of functions. Locate the set of points x such that $f_i|O_x$ are linearly independent. This set is open.

Proof Idea: Convert the linear independence of f_is on the *orbit* to linear independence of columns constructed at the *point*. This is the matrix $D = d_{ij}$ with $d_{ij} = p_i.f_j$.

Ingredient: The fact that a function being zero on the *whole orbit* is captured by all derivatives being zero at *one point*. That is, complex analytic functions have Taylor expansions.

Limitation: In the form presented, this proof works only over $\mathbb{C}$.

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*Leave the proof details for later. Expected time without details: 3 minutes, with details: 5 minutes

Claim: Under certain conditions, a nontrivial k linear relation over an orbit $\implies$ nontrivial k linear relation over whole space (X)

Ingredients:

- Nontrivial linear relation over subset $\implies$ nontrivial linear relation over Zariski closure

- Function vanishes on the zero set of a "Galois closed" ideal $\implies$ Function is in ideal

Special fact: When k is algebraically closed, a "radical ideal" is a "Galois closed" ideal, courtesy **Hilbert's nullstellensatz**

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*Expected time: 2 minutes

†Blackboard: explain Galois correspondence

Setup: Let P be the Zariski cone of $J^{+}S$.

Claim: If $J^{+}S$ is a radical ideal, $k = \mathbb{C}$ and O is a dense orbit in P, nontrivial k linear relation in M over $O \implies$ nontrivial k linear relation over X.

Proof Idea: Apply the ingredients on the previous slide. Will become clearer when done on the blackboard.

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*Expected time: 5 minutes

†Blackboard: Intensive use

Summary: We have obtained sufficient conditions for S to be free over J.

Overall claim: Let $k = \mathbb{C}$, J^+S be a radical ideal, and O be a dense orbit in P. Then, if $M \cap J^+S = 0$, J linear independence equals k linear independence for elements in M.

Ingredients:

- $\mathbb{C}$ is algebraically closed and J^+S is a radical ideal $\implies$ J^+S is Galois closed.

- The Lie algebra structure of $\mathbb{C}$ was used in transferring an orbit related question to a point related question.

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*Expected time: 3 minutes

Setup: k, X, S, J, J^+S

Question: Let H be a module theoretic complemenet to J^+S in S. How does the map $\gamma_x : H \to S(O_x)$ look?

Partial answer: It is surjective. It is an "isomorphism" in case any function in H vanishing on P is zero.

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*Expected time: 2 minutes

†Blackboard: Write map names. Brief use

Recall: We already proved that if O_x is dense in P, the map γ_x is injective.

Definition: If O_x is dense in P, x is termed **regular**(defined). If the saturation of O_x under the $\mathbb{C}^*$ action is dense in P, x is termed **quasi regular**(defined).

Question: Is γ_x an isomorphism for quasi regular x (the way it is for regular x)?

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*Expected time: 1 minute

Proof ingredients:

- The set of regular points is Euclidean nonempty open (nonvanishing minors) and the saturation of O_x is dense. So they intersect.

- The set of x for which γ_x is isomorphism is closed under $\mathbb{C}^*$ action.

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*Expected time: 2 minutes

†Blackboard: Explain proof idea

More comments:

- A general notion of **Galois correspondence**

- Other questions about **finitely generated** and about **free**.

- A natural complement to $J^{+}S$, namely the space of **harmonic polynomials**.

$* \dagger$

*Expected time: infinity

†Blackboard: Arbitrary use