ANNOTATIONS OF KOSTANT'S PAPER

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Abstract. Here, I annotate Section 0 of Kostant's paper on "Lie Group Representations on Polynomial Rings".

1. The setup

1.1. Automorphisms of the polynomial ring.

Let's start with a commutative unital ring R and consider an R algebra S. Then an R automorphism of S is an automorphism of S that restricts to the identity map on R. The question I discuss here is: what is the structure of the R automorphism group of S when S is a free R algebra? That is, what is $Aut_R(R[x_1, x_2, \ldots x_n])$?

To specify any endomorphism of S fixing R, we need to describe where each x_i goes. Each x_i goes to a polynomial $p_i(x_1, x_2 \ldots x_n)$. Every such choice of polynomials gives a unique endomorphism. Thus the structure of $End_R(S)$ is simply the collection of all n tuples of polynomials with multiplication being composition. Note that $End_R(S)$ is a noncommutative monoid.

An **linear endomorphism**(defined) of $R[x_1, x_2 \ldots x_n]$ is an automorphism that sends each x_i to a linear combination of the x_is. The composite of two linear maps corresponds to the composite of the corresponding endomorphisms of $R[x_1, x_2 \ldots x_n]$. Thus, $M_n(R)$ (the monoid of linear maps on R^n) is a submonoid of $End_R(R[x_1, x_2 \ldots x_n])$.

A **linear automorphism**(defined) is thus an invertible linear endomorphism. The corresponding linear map lies in $GL_n(R)$. Thus, $GL_n(R)$ is a subgroup of $Aut_R(R[x_1, x_2 \ldots x_n])$.

What is so special about linear automorphisms?

1.2. The polynomial ring as a graded algebra.

Define the **degree**(defined) of a monomial as the sum of the indices of all the x_is. The degree of a polynomial is then the maximum of the degrees of all monomials with nonzero coefficient. A **homogeneous polynomial**(defined) is a polynomial in which all monomials with nonzero coefficients have equal degree. If this degree is d, the polynomial is said to be homogeneous of degree d.

Every polynomial can be broken down uniquely as a sum of homogeneous polynomials of degrees d, as $d \in \mathbb{N}_0$. The collection of all homogeneous polynomials of a given degree d, along with the zero polynomial, forms an R module. Thus, the ring $R[x_1, x_2 \ldots x_n]$ is a module theoretic direct sum of the modules of homogeneous polynomials of degree d. If S_d denotes the module of homogeneous polynomials of degree d, then:

- Each S_d is an R module.
- $S = R[x_1, x_2 \ldots x_n]$ is a direct sum of the S_ds.
- $S_d S_f \subseteq S_{d+f}$

For a general S such a collection of S_ds is termed a **gradation**(defined) and S along with its gradation is termed a **graded R algebra**(defined). If, further $S_0 = R$ (which is true in the polynomial case) S is termed a **connected graded R algebra**(defined). The component S_d is termed the **homogeneous component**(defined) of degree d.

A **graded algebra automorphism**(defined) of S (as a graded R algebra) is an algebra automorphism of S that restricts to a module automorphism on each homogeneous component of S. In other words, it is a degree preserving automorphism. We can now readily check:

Claim. The graded algebra automorphisms of $S = R[x_1, x_2 \ldots x_n]$ with the usual gradation are precisely the linear automorphisms.

1.3. A general Galois correspondence.

1.4. **What it translates for GL_n.** Here, I set the scope for the Kostant paper's section 0.

We will look at the polynomial ring *only* over a field, and we will usually assume the field to be $\mathbb{C}$. And we will *only* consider subgroups of GL_n. Some consequences are:

2. THE SETUP AND TERMINOLOGY

2.1. **Notation.** This notation shall be used throughout Kostant's paper:

G: subgroup of $GL_n(k)$
X: vector space k^n
S: ring $k[x_1, x_2 \dots x_n]$
J: fixed points of S under G action
J^+: elements of J with 0 constant term

For instance if $G = GL_n(k)$, then the only fixed points are the constant polynomials, so $J \cong k$. And J^+ is the trivial ring.

2.2. **Computing fixed polynomials.** Given a subgroup of $GL_n(R)$, how do we compute its ring of invariant (fixed) polynomials?

Here are the steps:

- First, determine the orbits of $X = k^n$ under the action of G. Note that the invariant polynomials are dependent only on the orbits. So, the Galois closure of G contains all the elements that define the same orbits on X.
- Now, for each orbit, determine the corresponding ideal of polynomials that vanish on the orbit. Let this be I. The ring of polynomials that is constant over the subring is then $I + k$, because k is the base ring.
- Take the intersection of these rings $I + k$ over all orbits.

Let us work out some examples.

(1) The group SL_n is transitive on k^n. Thus, its closure in GL_n is the whole of GL_n. When k is an infinite field, the ideal of functions that vanish on the whole of k^n is the trivial ideal. The ring of functions constant on the whole of SL_n is thus the ring of constant polynomial functions – a copy of k itself.
(2) The group SO_n has, as its orbits, concentric spheres centered at the origin. Its closure with respect to orbits is O_n. The ideal of functions vanishing on each sphere is precisely the ideal generated by $\sum_i x_i^2 - r^2$ where r is the radius of the sphere. Thus, the ring of functions constant on a given sphere is the direct sum of the corresponding ideal and k. The intersection of all these rings is the subring generated by the polynomial $\sum_i x_i^2$. The proof just involves some clever reasoning.
(3) The group S_n has orbits of finite size. Points in the same orbit have the same coordinates upto permutation. By a result of Newton (without going through the orbit wise analysis) we have that the ring of invariant functions is the polynomial ring in the elementary symmetric functions. This is isomorphic to the whole of $k[x_1, x_2 \dots x_n]$ and has transcendence degree $n!$.

2.3. **The invariant polynomials without constant term.** The method I outlined above to determine the invariant subring of a subgroup of GL_n was: determine the orbits, find the ideal of polynomials vanishing on each orbit, and the ring of constant polynomials and take the intersection of all the subrings. The subring J that we get after taking the intersection definitely contains the ring of constant polynomials – k. In fact, k is a *retract* of J via the map taking each polynomial to its constant term. The kernel of this retraction is J^+, the subset of J containing invariant polynomials with zero constant term. J^+ in itself may not be an ideal, but J^+S is an ideal.

A nice question: what is the structure of J^+S? Let's look at the cases studied so far:

- When $G = GL_n$ or SL_n then $J = k$ and J^+S is the zero ideal.
- When $G = O_n$ or SO_n then J is the subring generated by the sum of squares polynomial. J^+S is the principal ideal generated by $\sum_i x_i^2$.
- When $G = S_n$ then what?

2.4. **A complement to J^+S.** Let H be a graded subspace complementary to J^+S in S. Then, Kostant claims that it is easy to see that:

Claim.

$$S = JH$$

Proof. We have $S = J^+S + H$. Note that $H = kH \subseteq JH$.

Further, $J^+S \subseteq JS = J(J^+S + H)$ □

Let's first look at the examples:

- When $G = GL_n$ or SL_n then J^+S is trivial so H is the whole of S. And $J = k$. So clearly $S = JH$.
- When $G = O_n$ or $G = SO_n$ then J^+S is the principal ideal generated by the sum of squares polynomial. I'm still trying to figure out a complement to it.
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2.5. Interpretation via harmonic polynomials.

2.6. Tensor products and freeness. In our setup, J is a subring of S. The relationship between S and J is rather strange – S is rarely a polynomial algebra over J. However, it often happens that as a module, S is J free. Kostant claism the following:

Claim. S is J free if and only if $S = J \otimes H$.

3. Looking at the orbits

On my own, I had described the approach towards determining the ring of invariant polynomials under a given subgroup of GL_n. Now, Kostant discusses the same idea. He looks at the orbit O_x of a vector x, and consider $S(O_x)$ to be the ring of functions on O_x obtained by restriction from S. $S(O_x)$ is the quotient ring of S by the ideal of polynomials vanishing on O_x.

Kostant goes a bit further. He observes that in fact, the whole of J maps to k under the above quotient map, so H must map to the whole of $S(O_x)$.

3.1. Getting to the algebraic geometry. We know what's a Lie group, and what is the associated Lie algebra. In essence, the Lie algebra is the tangent space at the origin to the Lie group. The action of the Lie group on the Lie algebra is termed the adjoint action. We'll study the following setup:

- k is the field of complex number $\mathbb{C}$.
- X is the underlying vector space of a Lie algebra.
- G is the corresponding Lie group with the adjoint action on X.

Here are some possibilities for G and X:

- $G = GL_n$ and $X = M_n$ with the Lie bracket defined as the usual matrix theoretic commutator.
- $G = SL_n$ and X is the collection of trace 0 matrices.
- $G = SO_n$ and X is the collection of skew symmetric matrices.

3.2. The rational functions. Given a variety Y, we can consider the everywhere defined rational functions on Y. This is a ring I call $R(Y)$. Clearly for O_x, we have

$$S(O_x) \subseteq R(O_x)$$

Note also that $R(G/G^x)$ is isomorphic to $R(O_x)$

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