

# INVARIANT THEORY: BASED ON KOSTANT

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ABSTRACT. I've always been fascinated by the problems of invariant theory. I'm currently reading Kostant's paper for VSRP under Professor Dipendra Prasad. I also stumbled on a book by Igor V. Dolgachev on Invariant Theory. Based on these, and my own understanding gathered from other sources, I document what I think invariant theory is and could be about.

## 1. GROUPS ACTING ON SETS

1.1. **A basic Galois correspondence.** Let  $A$  and  $B$  be two sets, with a relation  $R$  between  $A$  and  $B$ . That is,  $R$  is a map  $A \times B \rightarrow \{0, 1\}$ , with  $a$  related to  $b$  if and only if  $R(a, b) = 1$ . Then, define  $f : 2^A \rightarrow 2^B$  as follows. For  $S \subseteq A$ ,  $f(S)$  is the set of all elements  $b \in B$  such that  $R(a, b) = 1$  for every  $a \in S$ . Similarly  $g : 2^B \rightarrow 2^A$  takes  $T \subseteq B$  to the set of all elements  $a \in A$  such that  $R(a, b) = 1$  for every  $b \in T$ .

The following are true:

- If  $S_1 \subseteq S_2 \subseteq A$ , then  $f(S_2) \subseteq f(S_1)$ . Analogously for  $B$  and  $g$ .
- For any  $S \subseteq A$ ,  $S \subseteq g(f(S))$ . Analogously for  $B$  and  $g$ .
- Combining the above two,  $f(g(f(S))) = f(S)$ .

Subsets in  $B$  of the form  $f(S)$  are closed under the operation  $f.g$  while subsets of  $A$  of the form  $g(T)$  are closed under the operation  $g.f$ . Moreover, for any given subset of  $A$ , its closure is  $g.f$  applied to it. This "closure" satisfies all the conditions of a closure operation, namely:

- Closure of the closure is the closure. A set which is its own closure is a closed set.
- An arbitrary intersection of closed sets is closed.

In this article, we shall talk of two instances of Galois correspondence:

- The *fixed point relation* when a group acts on a set. (refer 1.2)
- The *vanishing relation* for a set and a ring of functions from that set to a field. (refer 2.2)

1.2. **For group actions.** When a group acts on a set, we consider the "fixed point" relation between the group and the set. That is, an element of the group is said to be related to an element of the set if the group element fixes the set element via its action. If a group  $G$  acts on a set  $S$  via an action  $\cdot$ , then  $g \in G$  is related to  $s \in S$  if  $g.s = s$ .

Any closed subset of the group is a subgroup, and is termed a **fixing subgroup**<sub>(defined)</sub> and any closed subset of the set is termed a **fixed point subset**<sub>(defined)</sub>.

1.3. **Derived actions.** Let's say we want to study a nice group, say a finite group (just given purely as a group) or a Lie group (given with a topological, analytic structure). We have a huge arsenal of purely group theoretic techniques to handle it. But we often try to *represent* the group as acting on a set. We extract the maximum information when we choose this set to be an Abelian group, a module over a ring, or a vector space over a field, and require the group action to be via automorphisms of that structure.

The representation on the vector space itself gives lots of interesting data: the character map, the determinant map etc. as well as some orthogonality relations that help us determine the possible representations effectively. However, we gain even more insight by obtaining other *derived* actions from this action – such as the following:

- A group acting as automorphisms on a vector space also acts on the collection of subspaces of a the vector space of a given dimension. The manifold whose points are  $r$  dimensional subspaces of an  $n$  dimensional space is termed the **Grassmannian manifold**<sub>(defined)</sub> and is denoted as  $Gr(n, r)$ . The group thus acts as automorphisms of the Grassmannian.
- A group acting as automorphisms on a vector space also acts on the collection of ordered bases of  $r$  vectors. This is termed the **Stiefel manifold**<sub>(defined)</sub>.

- A group acting as automorphisms on a vector space also acts as automorphisms on each of its tensor products with itself. In particular, it acts on the **tensor algebra**<sub>(first used)</sub>. It also acts on the **symmetric tensor algebra**<sub>(defined)</sub>, which is the quotient of the tensor algebra by commutation relations, and on the **alternating tensor algebra**<sub>(defined)</sub>, which is the quotient of the tensor algebra by anticommutation relations.

1.4. **What we do with representations.** We could try looking for fixed points. This boils down to the approach discussed in subsection

- When given a representation (or action) of a group, try to look for a **subrepresentation**<sub>(first used)</sub>. A subrepresentation is essentially the action of the group on a subset that is invariant as a whole. When studying linear representations alone, we are interested in linear subrepresentations. The famous **Maschke's averaging lemma**<sub>(result assumed)</sub> tells us that any linear representation of a finite group on a finite dimensional vector space can be uniquely decomposed as a direct sum of linearly irreducible representations.

This aspect of the study comes under the broad heading of *module theory*<sub>(topic name)</sub> and *representation theory*<sub>(topic name)</sub>.

- After breaking it down into linearly irreducible representations, try looking at *invariant varieties*. These are unions of orbits that collectively form a **Zariski set**<sub>(first used)</sub>.

This part of the study comes under the broad heading of *algebraic geometry*<sub>(topic name)</sub>.

1.5. **Deeper understanding of the tensor product action.** Let  $V$  be a vector space over a field  $k$ . Then  $GL(V)$  is the group of all automorphisms of  $V$ . Clearly, every such automorphism gives an automorphism on the tensor algebra of  $V$ , namely, by acting on pure tensors coordinate wise and extending linearly.

The tensor algebra over a vector space is the classical example of a graded algebra:

**Definition.** Given an algebra  $S$  over a ring  $R$ , a **gradation**<sub>(defined)</sub> of  $S$  over  $R$  is a collection  $S_m$  ( $m \in \mathbb{N}_0$ ) of  $R$  submodules of  $S$  such that:

- $S$  is the direct sum of all  $S_m$ s, that is

$$S = \bigoplus_m S_m$$

- $S_m S_n \subseteq S_{m+n}$

$S$  along with its gradation is termed a **graded algebra**<sub>(defined)</sub> over  $R$ . If  $S_0 = R$ ,  $S$  is termed a **connected graded algebra**<sub>(defined)</sub>.  $S_m$  is termed a **homogeneous component**<sub>(defined)</sub> of degree  $m$ .

The tensor algebra of  $V$  is a connected graded algebra with  $S_m$  being the tensors of rank  $m$  and  $S_0$  being the base field  $k$ . We now need to be careful while using the term *automorphism* for a tensor algebra because automorphism could mean either of two things:

- **algebra automorphism**<sub>(defined)</sub>: This may or may not preserve the gradation. It only preserves the additive and multiplicative structure. An algebra *endomorphism* is defined by arbitrarily specifying where a basis for  $V$  goes. For it to be an algebra *automorphism* we must ensure its invertibility. Examples of algebra automorphisms of the polynomial ring are *triangular* automorphisms, which send each  $x_i$  to  $x_i$  plus a polynomial in the  $x_j$ s for  $j < i$ .
- **graded algebra automorphism**<sub>(defined)</sub>: This definitely preserves the gradation. In particular, it restricts to a linear automorphism on  $V$  itself (the homogeneous component of degree 1). Moreover, because the effect of the automorphism on  $V$  gives its effect everywhere, we conclude that the graded algebra automorphisms of the tensor algebra are precisely the linear automorphisms of the vector space. We shall concentrate on the graded algebra automorphisms of the tensor algebra and its quotients.

1.6. **The overall programme.** The programme is as follows:

- Start with a group acting on an Abelian group, a module, or a vector space: One way is to look at subgroups of  $GL_n(R)$  acting on  $R^n$ . This can be done for any commutative unital ring  $R$ . Another is to look at subgroups of  $GL_n(\mathbb{R})$  or  $GL_n(\mathbb{C})$  acting on their Lie algebras. This can also be extended, in some cases, to commutative unital rings in general, by defining a *formal* analogue of the Lie algebra. I'm not really sure about that, though.
- Use this to obtain an action of the group on an algebra e.g. the tensor algebra, the exterior algebra (another name for alternating tensor algebra), or the symmetric algebra.

- Study the fixed point subalgebra of this algebra. There are general theorems and we can also work out the special details.
- Understand the relation between the base ring, the fixed point subalgebra, and the whole algebra.
- Try to decompose the representation in terms of its linearly irreducible subrepresentations, go further and study the indecomposable varieties, and so on.

## 2. DETERMINATION OF INVARIANTS

**2.1. Determining invariant functions.** As in the previous subsections, let  $k$  be a field and  $V$  be any set (it could be an affine space, a projective space, a variety). Let  $S$  be a  $k$  algebra of functions  $V \rightarrow k$ . Clearly,  $S$  contains all constant functions. Suppose  $G$  is a group acting on  $V$ . Question: how does  $G$  act on  $S$ ? The usual action is  $(g.s)(x) = s(g^{-1}(x))$ . This means that, for  $s \in S$ , we calculate  $s$ , not at  $x$ , but at the point which will come to  $x$  under the action of  $g$ .

An *invariant function* is then a function  $s \in S$  such that  $s = s.g^{-1}$  for every  $g \in G$ . This is equivalent to requiring the value of  $s$  to be constant on each orbit in  $V$  under  $G$  action. Some immediate conclusions:

- If two groups  $G_1$  and  $G_2$  define the same orbits on  $V$  then the ring of invariant functions is the same. Thus the closure of a group with respect to the Galois correspondence with invariant functions, at the very least contains all elements that preserve orbits for this group.
- For an orbit  $O$ , let  $I_O$  be the ideal of functions in  $S$  that vanishes on  $O$ . Then the subring  $R_O$  of functions constant on  $O$  is precisely  $I_O + k$  where  $k$  here denotes the ring of constant functions. The ring of invariant functions under the action of  $G$  is

$$\bigcap_O R_O = \bigcap_O (I_O + k)$$

**2.2. A Galois correspondence of vanishing.** Carrying forward the notation of the previous subsection,  $V$  is a set,  $k$  a field, and  $S$  a  $k$  algebra of functions  $V \rightarrow k$ . We define a relation between  $S$  and  $V$  by the rule:  $s \in S$  is related to  $v \in V$  if  $s(v) = 0$ , that is,  $s$  vanishes on  $v$ . (Note that the image here is in  $k$ ).

The associated Galois correspondence takes a subset of  $V$  to the set of functions (viz elements of  $S$ ) vanishing on that subset. We know that any such subset must be a radical ideal. The other way around, it takes a subset of  $S$  and gives the set of all points in  $V$  where all those functions vanish. This is the **zero set**<sub>(defined)</sub> of the subset of  $S$ .

Some observations:

- Every closed element in the ring of functions is a radical ideal. The converse statement (that every radical ideal is a closed set) holds true for the polynomial ring when the underlying field is algebraically closed. This statement is the substance of **Hilbert's nullstellensatz** (result recalled).
- The above Galois correspondence describes a closure operation on the set  $V$ . In our case, a finite union of closed sets is closed because we can take the *product* of functions vanishing on each. Thus, the Galois correspondence in this case actually defines a topology on  $V$ .

A quick step back and survey. Suppose we have a group  $G$  acting on a set  $V$ , and an algebra  $S$  of functions  $V \rightarrow k$ . We want to correlate three things:

- The *orbit decomposition* in  $V$  under  $G$ . Translate: What does the orbit of each point look like? What are the sets in  $V$  that arise as unions of orbits?
- The *vanishing correspondence* between  $S$  and  $V$ . Translate: What are the closed sets in  $V$  under  $S$  action and how does the topology look? What are the closed ideals with respect to the correspondence?
- The *fixed point correspondence* between  $G$  and  $S$ . Translate: Which functions in  $S$  are  $G$  invariant?

So far, to determine the fixed point correspondence, we: first determine the orbit decomposition, then use the vanishing correspondence to find all the constant subrings on each orbit, and then intersect these. We had seen that what really matters in a group action are its orbits. Now, we go a step further: what really counts is the closure of the orbit with respect to the vanishing correspondence, and not the orbit itself. This is because the radical ideal of functions vanishing on the *closure* of the orbit is equal to the radical ideal of functions vanishing on the orbit.

More on this soon.

**2.3. The symmetric tensor case.** Here's a map from the tensor algebra to the ring of functions  $k^n \rightarrow k$  by the following evaluation map: for a given tuple  $(v_1, v_2 \dots v_n)$  replace  $x_i$  by  $v_i$ , replace the tensor product operation by the usual multiplication, and the formal addition by the field addition. This map is a homomorphism of  $k$  algebras.

Clearly, the map is not injective, because  $x \otimes y$  and  $y \otimes x$  give rise to the same function:  $xy$ . That's because all tensors whose symmetrization is zero give the zero function. Thus, we can view this as a map from the symmetric tensor algebra to the algebra of functions. Note that the target is a commutative algebra, so this is the best we can hope for. The map is simply the evaluation map for polynomials.

The question: is the map from the symmetric tensor algebra to the algebra of functions injective? The answer is *yes*. In other words, any polynomial (over an infinite field  $k$ ) that takes the value zero at all points in  $k^n$  is the zero polynomial.

The embedding of the symmetric tensor algebra in the algebra of functions allows us to apply the technique of the previous subsection to compute the invariant subrings.

The symmetric tensor algebra (or polynomial algebra) situation can be *somewhat* generalized. Suppose  $S$  is a commutative  $k$  algebra and  $V$  is the collection of maximal ideals of  $S$  having quotient  $k$ . Clearly, there is a map  $S \times V \rightarrow k$  that sends each pair  $(s, v)$  to the quotient of  $s$  under the maximal ideal  $v$ . We can thus think of each  $s \in S$  as inducing a function  $V \rightarrow k$ , and we can readily verify the conditions for it to be an algebra of functions. Then, we can use the dictionary and techniques of the previous subsection.

In our case,  $V = k^n$ , and  $S$  is the polynomial ring in  $n$  variables with the corresponding action being the *evaluation map* from  $k^n$  to  $k$ . This is a special case of the general situation outlined above, because by the *weak form* of **Hilbert's nullstellensatz** (result assumed) points in  $k^n$  can be identified with maximal ideals having quotient  $k$ .

Moer on this as we proceed.

We begin with a general theorem:

**Theorem 1.** If the orbits under the action of a group on the  $k$  vector space  $V$  are all of the form  $p(x_1, x_2 \dots x_n) = c$  with  $p$  a fixed polynomial and  $c$  a parameter, then the invariant subring under the group action is the subring generated by the polynomial  $p$ .

*Proof.* Suppose  $q(x)$  is a polynomial constant on each orbit  $p(x) = c$ . Take any nonempty orbit. Then,  $q(x)$  must be a constant  $c_1$  plus an element in the ideal generated by  $p(x) - c$ . That is:

$$q(x) - c_1 = q_1(x)(p(x) - c)$$

Note that both  $q(x) - c_1$  and  $p(x) - c$  are invariant on every orbit, and hence, so is their ratio. Thus  $q_1(x)$  also lies inside the invariant subring. If, by induction on the degree, we assume that  $q_1(x)$  is in the polynomial subring generated by  $p(x)$  then we obtain that  $q(x)$  is also in the polynomial subring.  $\square$

**Corollary 1.** Let  $G$  be a subgroup of  $GL_n(k)$  obtained as the set of linear transformations  $x$  such that  $axa^T = a$  for a given symmetric nonsingular matrix  $a$ . That is,  $G$  preserves a nondegenerate bilinear form. Then, the orbits on  $V = k^n$  are of the form  $p(x) = c$  where  $p$  is a quadratic polynomial and  $c$  is a parameter for the orbit. Hence, the invariant subring under the group action is a subring generated by a quadratic polynomial.

In the next subsection, we shall see two special cases of this:  $O_n(k)$  and  $Sp_{2n}(k)$ .

#### 2.4. Some invariant polynomial subring computations.

- (1)  $GL_n(k)$  on  $k^n$ : Here, there are two orbits – the fixed point 0 and the rest of the space. The same is true for  $SL_n$ , so  $GL_N$  is the orbit closure of  $SL_n$ . The only polynomials constant on the whole of  $k^n \setminus 0$  are the constant polynomials, and this is the invariant subring.
- (2)  $O_n(k)$  on  $k^n$ : Here, the orbits are spheres, viz solutions to polynomial equations of the form  $\sum_i x_i^2 = c$  for a constant  $c$ . The invariant subring is thus the polynomial subring generated by the sums of squares. A polynomial in this subring is termed a **radical polynomial**<sub>(defined)</sub>.
- (3) Action of  $Sp_{2n}(k)$  on  $k^{2n}$ : Here, the orbits are of the form  $\sum_{i=1}^n x_i^2 - \sum_{j=n+1}^{2n} x_j^2 = c$ . The invariant subring is thus the polynomial subring generated by  $\sum_{i=1}^n x_i^2 - \sum_{j=n+1}^{2n} x_j^2$ .
- (4) Action of  $B_n$  on  $k^n$ : Here  $B_n$  is the group of upper triangular matrices of nonzero determinant. The orbits under this action are parameterized by the *leftmost nonzero coordinate*. In other words, there is one orbit where the first coordinate is nonzero, one orbit where the first coordinate is zero but the second coordinate is nonzero, and so on, culminating in the fixed point 0 which is

the orbit with all coordinates 0. Orbits under the  $B_n$  action on  $k^n$  are not closed in the Zariski sense, but they are locally closed. Because one of the orbits is Zariski dense, we conclude that the only invariant polynomials are the constant polynomials.

**2.5. What happens without symmetric?** Can we embed the tensor algebra in an algebra of functions? The basic problem: the tensor algebra is noncommutative. Thus, in the sense of a function space to  $k$ , the best we can do is the evaluation map described above. However, what we can try to do in the general case is to view *each homogeneous component* of the tensor algebra as a space of functions and deduce its invariants.

- (1) Action of  $GL_n(k)$  on  $k^n$ : I believe it is true that the only elements of the tensor algebra invariant under  $GL_n$  action are the constant tensors.
- (2) Action of  $O_n(k)$  on  $k^n$ : This needs to be figured out somewhat. Clearly, it is in the inverse image of the polynomial subring generated by  $\sum_i x_i^2$ . Can we say anything better?

In the above analysis, we somehow make use of the fact that all coordinate permutations are contained in  $S_n$ .

I don't really know how to do this, and besides, it's not really part of the paper, so I'll fill this in later.

### 3. THE RING THEORY OF INVARIANTS

**3.1. It would be nice if..** In subsection 1.6, we had decided that the overall programme here is to start with a representation of a group, obtain more related representations, and then try to compute *fixed points* and *invariant subsets* and use these to obtain *decompositions*. In the previous section, we took a quick look at how we can start with a subgroup of  $GL_n(k)$  acting on  $k^n$ , make it act on the polynomial ring, and compute the ring of invariant polynomials.

Kostant's paper is largely about the study of the polynomial ring. But for a little time, we'll step back and understand a general action of a group on an algebra over a ring.

**3.2. New setup.** Let  $S$  be an algebra over a ring  $R$ . Let's say we have an action of a group  $G$  on  $S$ , leaving an invariant subring  $J = S^G$ . Question: How does the nature of  $G$  govern the ring theoretic relationship (or  $R$  algebra theoretic relationship) between  $S$  and  $S^G$ ?

The larger the group  $G$ , the smaller  $S^G$  with respect to  $S$ . Thus the size of  $S$  over  $S^G$  should reflect the size of  $G$ . The best thing in the world is if we can naturally identify  $S/S^G$  with  $R[G]$  with the action of  $G$  being the regular representation.

The problem with the base ring being arbitrary is that by and large, modules over rings need not be free. Thus, meaningful theorems can be proved only when the base ring  $R$  is a field (in which case all  $R$  modules are free). So, we use field notation for this base ring, by calling it  $k$ .<sup>1</sup>

**Claim.** Let  $G$  be a finite group of automorphisms acting on a  $k$  algebra  $S$ . Then  $S$  is an integral extension of the invariant subring  $S^G$ , with degree at most the cardinality of  $G$ .

*Proof.* Take an element in  $S$ . Look at its orbit under  $G$ . Clearly all the symmetric functions of the elements in this orbit are in  $S^G$ . Thus, the element in  $S$  is the root of a monic polynomial in  $S^G$ . Moreover, the degree of the polynomial equals the size of the orbit, which is at most the cardinality of  $G$ . □

**Claim.** Let  $G$  be a finite group of automorphisms of a finitely generated  $k$  algebra  $S$ . Then the subalgebra  $S^G$  is finitely generated over  $k$ .

*Proof.* Let  $J = S^G$ . As shown above,  $S$  is integral over  $J$ , of finite degree. Let  $J'$  be the subalgebra of  $S$  (in fact of  $J$ ) generated by the coefficients of the monic polynomial whose roots are the generators of  $S$ . We first claim that  $J'$  is Noetherian. This follows essentially because  $J'$  is a free algebra over  $k$  in a certain number of variables. Then, since  $a$  is a finitely generated  $J'$  module, so is  $J$ . Since  $J'$  itself is finitely generated over  $k$ , we have that  $J$  is finite generated over  $k$ . □

The key point of the proof is to find a Noetherian subring of  $J$  which is finitely generated over  $k$  as an algebra, and over which  $S$  is finitely generated as a module.

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<sup>1</sup>An interesting question would be: would the proofs still go through if we assumed  $R$  as a general ring but explicitly assumed that the relevant modules were  $R$  free?

#### 4. GETTING DOWN TO KOSTANT'S SETUP

**4.1. Preliminary notation and setup.** I introduce here the basic notation and setup that Kostant follows in his paper. Of course, there's this point that we shouldn't get so caught up in the symbols that we lose sight of what they represent. So, at all places I'll try to give the statement in two ways: one a *symbol free* way, and the other, in terms of the symbols.

For various levels of generality:

- Algebra over a ring:  $R$  denotes the base ring and  $S$  denotes the algebra over  $R$ .
- Algebra over a ring where the base ring is a retract:  $R$  denotes the base ring,  $S$  denotes the algebra, and  $S^+$  denotes the ideal complement to  $R$ . For a submodule  $N \leq S$ ,  $N$  is termed a BIR submodule if  $N$  is the direct sum of  $N \cap S^+$  and  $R$ . In that case  $N \cap S^+$  is denoted as  $N^+$ .
- Connected graded algebra over a ring:  $R$  denotes the base ring,  $S$  denotes the algebra, and  $S_m$  denotes the  $m^{\text{th}}$  homogeneous component of  $S$ . It is naturally a BIR algebra so the terminology for BIR algebras is applicable here. Graded submodules are BIR submodules as well.
- Working over a field: Replace the  $R$  for base ring with a  $k$  for the base field.
- The polynomial algebra (for a field): Let  $X$  denote the  $n$  dimensional vector space over  $k$ . Then the polynomial ring in  $n$  variables is the symmetric tensor algebra of  $X$  over  $k$ . We denote this by  $S$ . The remaining conventions follow the setup of a graded algebra.

By and large, we shall stick to the last setup: a field  $k$ , a vector space  $X = k^n$ , the polynomial ring  $S$ , a group  $G$  acting on  $X$  linearly,  $J$  is the ring of invariants, and  $N^+$  for a graded submodule denotes the sum of homogeneous components of positive degree.

**4.2. How differential operators come in.** In the short article on subrings and ideals, I outlined a general approach of looking at algebras where the base ring is a retract, and made some observations about such algebras. I also discussed the special case of freeness. This setup arises very naturally in the context of invariant theory. This is how.

Suppose we have a symmetric nonsingular bilinear form on the polynomial ring  $S$  over a base field  $k$ . Under this form, we can obtain orthogonal complements in terms of modules. A natural form occurring in polynomial rings is the differentiation form.

We first introduce the algebra of operators, namely:  $k[\frac{\partial}{\partial x_i}]$  where  $1 \leq i \leq n$ . Each element of the ring acts as a differential operator on the space of polynomials, the sum of two elements is the pointwise sum and the product is the composite differential operator. The addition clearly forms an Abelian group, and multiplication is clearly associative. Commutativity of multiplication is essentially the polynomial case of **Fubini's theorem** (result assumed) and distributivity follows from the fact that differential operators are linear maps.

The algebra of polynomials is isomorphic to the algebra of differential operators, with the obvious choice of isomorphism sending  $x_i$  to the partial derivative  $\frac{\partial}{\partial x_i}$ . From this view, the action of the differential operator on the polynomial ring turns into a  $k$  bilinear map. We can check that the map is nonsingular and can invoke the existence of orthogonal polynomials.

However, the ring of differential operators does not commute with the ring of usual polynomials, where each polynomial is treated as the corresponding multiplication map. So here are some definitions:

**Definition.** Let  $G$  be a group acting on the ring of differential operators. A polynomial, viz an element of  $k[x_1, x_2 \dots x_n]$  is said to be **harmonic**<sub>(defined)</sub> with respect to  $G$  if its derivative is zero under any differential operator of positive degree that is invariant under  $G$ .

Kostant now makes a basic asserion viz:

Let  $S$  be the whole polynomial ring,  $J^+$  be the set of invariant polynomials without constnat terms, and  $H$  be the set of ring of harmonic polynomials. Kostant claims that:

$$S = J^+ S + H$$

Here's how we understand this claim: These are orthogonal complements with respect to the given bilinear map.

Now, we can invoke results discussed in the other text to conclude that  $S$  is free over  $J$  if and only if  $S = J \otimes H$  with the tensor product collapsing to multiplication in  $S$ .

**4.3. Specific examples of these.** Here are the usual examples:

- (1) The full rotation group  $O(n)$  acting on  $k^n$ : We discussed earlier that the invariant polynomials form a subring called the ring of radical polynomials, which are simply polynomials in  $\sum_i x_i^2$ . The

harmonic polynomials for this group action are the polynomials whose Laplacian is zero. This is justified by the fact the corresponding to the polynomial  $\sum_i x_i^2$ , the corresponding Laplacian is  $\sum_i \partial^2 \partial x_i^2$

In the case of the rotation group, it turns out that indeed  $S = J \otimes H$ . This is also known as the **separation of variables theorem** (result stated), which, in simple terms, states that every polynomial can be expressed as finite sum of products of radical and harmonic polynomials.

- (2) Chevalley proved that when  $G$  is a finite group generated by reflections, then  $S = J \otimes H$ . Moreover,  $H$  itself is isomorphic to  $kG$  with the  $G$  action being the regular representation. In particular, this is true for the symmetric group  $S_n$ .

**4.4. The ring of functions.** The term “polynomial map” makes sense in a vector space. Question: what are the properties that the restrictions of these polynomial maps, when applied to subsets, possess?

Algebraic geometry has the answer. Namely, given any Zariski closed set, the collection of functions obtained by restricting polynomials in the vector space to the closed set is the quotient of the polynomial ring by the corresponding radical ideal. This quotient is called the **coordinate ring** (defined) of the Zariski closed set. When the Zariski closed set is viewed as an abstract entity, it is termed an **affine variety** (defined).

There’s also a notion of the ring of rational functions on a variety. These are ratios of polynomials on the whole vector space such that the denominator never vanishes within that subset.

Sometimes, the subsets we take may not be varieties (that is, they may not be Zariski closed). We can *still* talk of the restrictions of polynomial functions but we cannot make sense of the notion of *rational functions*.

There are a few more points:

- In the case of  $\mathbb{R}$  or  $\mathbb{C}$  and subsets of the vector space over these, there are two topologies – the usual **Euclidean topology** (defined) and the **Zariski topology** (defined). The Euclidean topology is much finer than the Zariski topology.<sup>2</sup> Thus, closed sets in the Euclidean sense may not be varieties.
- An interesting question (which we shall ponder a lot as we proceed) is: when do restrictions of polynomials give all the rational functions?
- Another question that’s equally important is: is there a retract subring? A retract module? For it.

**4.5. The “nice” situations.** Let’s recall the basic aim: Look at a group of linear transformations acting naturally on the polynomial ring and then proceed to understand the structure of the representation through techniques like: understanding it as an algebra over the subring of invariants, breaking it into irreducible components, etc.

To summarize, Kostant wants to know *when* these three nice things happen:

- (1) The ring of all polynomials is module theoretically free over the ring of invariant polynomials. In our jargon, it translates to  $S = J \otimes H$ .
- (2) For certain orbits, and every function on that which is obtained through restricting a polynomial to that orbit, there is precisely one harmonic polynomial which restricts to that function. In symbols, the map  $H \rightarrow S(O_x)$  is an isomorphism for certain  $x \in X$ . Another way of saying this is that  $H$  is a module theoretic complement to the ideal of functions corresponding to an orbit.
- (3) When the above holds, the functions obtained by restricting polynomials are in fact all the rational functions on  $O_x$ . In symbols,  $S(O_x) = R(O_x)$ .

The rotation group is an example where all the three hold.

Property spaces we could use these notions for:

- Representations of arbitrary groups in vector spaces: Let  $G$  be a group acting (without loss of generality) faithfully on  $k^n$ . Obtain the induced action on the polynomial ring and then ask questions 1, 2 and 3. Each of these conditions gives a property that a given group representation may either have or not have. That is, given a group  $G$  and a representation of  $G$  on  $k^n$ , it either satisfies condition 1 or doesn’t, it either satisfies condition 2 or doesn’t, and it either satisfies condition 3 or doesn’t.
- Representation of a Lie group on the associated Lie algebra: let  $G$  be a (real or complex) Lie group. Consider the natural action of  $G$  on its connected component. We can again ask questions

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<sup>2</sup>Equivalent to the observation that polynomial functions are continuous

1, 2 and 3. Thus, given a Lie group, it either satisfies condition 1 or doesn't. It either satisfies condition 2 or doesn't. It either satisfies condition 3 or doesn't.

**4.6. Sufficient conditions for each.** When do conditions 1, 2 and 3 hold? Kostant provides sufficient conditions for each. He also discusses how, for given situations, we can try to verify whether those sufficient conditions hold.

We first introduce another piece of notation: Let  $P$  be the cone of common zeroes of the ideal  $J^+S$ .

**Claim** (Sufficient condition for 1). The ideal corresponding to  $J$ , viz  $J^+S$ , is a prime ideal, and there is an orbit  $O_x$  which is dense in  $P$ .

Call an element  $x \in X$  quasi regular if  $P \subseteq \mathbb{C}^*O_x$ .

**Claim** (Sufficient condition for 2). *If the sufficient conditions described above for 1 hold, then 2 holds for all quasi regular elements  $x \in X$ .*

**Claim** (Sufficient condition for 3).  *$J^+S$  is prime, the closure  $\overline{O_x}$  is a normal variety and the set  $\overline{O_x} - O_x$  has codimension at least 2 relative to  $\overline{O_x}$ .*

## 5. JUSTIFICATIONS AND WORKINGS OUT

**5.1. Notation.** Now, we get into the thick of Kostant's paper. At the beginning of the preceding section I had introduced some general terminology. Now we are nearing the time when generalities need to be dispensed with and we need to work with specific fields. So it's time to free notation. Again, not something I like, but Kostant does it everywhere, so I'm just following him for convenience. I may rework the details later. First, a quick tabulation of the terminology and conventions:

Letter	Meaning
$n$	A positive integer
$X$	$\mathbb{C}^n$
$S$	polynomial ring over $X$
$G$	subgroup of $GL_n(X)$
$J$	ring of $G$ invariant polynomials
$J^+$	the positive degree elements of $J$
$H$	space of $G$ harmonic polynomials
$P$	vanishing set of $J^+$
$\mathfrak{g}$	Lie algebra of $G$ , same as $X$
$U$	universal enveloping algebra of $\mathfrak{g}$
$x$	some point in $X$
$O_x$	orbit of $x$
$G^x$	stabilizer of $x$ in $G$

Keep referring to this.

**5.2. A little on the Lie algebra perspective.** In our setup,  $G$  is a Lie group acting on its own Lie algebra via the **adjoint representation**<sup>(first used)</sup>. This is a very narrow special case of an arbitrary subgroup of  $GL_n(\mathbb{C})$  acting on  $\mathbb{C}^n$ . However, a lot of the interesting phenomena occur for these actions.

First, let's take  $G = GL_n(\mathbb{C})$  acting on  $M_n(\mathbb{C})$  (its Lie algebra) by matrix conjugation. This action is very different from the action of  $GL_n(\mathbb{C})$  on  $\mathbb{C}^n$ . What do the symbols interpret as in this case? My guesses:

$S$	all polynomials in $n^2$ variables
$J$	polynomials in the coefficients of the characteristic polynomial
$H$	can't put it in words
$P$	set of nilpotent matrices

The idea now is to get a qualitative feel of the orbits. Somewhere else, I observed that a conjugacy classes go to conjugacy classes under translation by central elements. I also built a little theory out of it. I want to revisit that theory... but for now, let's get on with what Kostant has to say.

**5.3. General case of a complex reductive Lie algebra.** Reductive means that the unipotent radical is trivial. I need to figure out what unipotent means for an abstract Lie algebra.

The **rank**<sup>(defined)</sup> of a reductive Lie algebra is the maximum number of simultaneously commuting linearly independent elements, or equivalently, the dimension of a **Cartan subalgebra**<sup>(defined)</sup> – a maximal Abelian subalgebra.

Chevalley gave a beautiful theorem which asserts that in for the adjoint representation of a Lie group  $G$  on its Lie algebra  $\mathfrak{g}$ , the subring of invariant polynomial ( $J$ ) is a polynomial ring in  $l$  homogeneous generators, with  $l$  being the rank of  $\mathfrak{g}$ .

The generators are  $u_i$  with the degree of each  $u_i$  being  $m_i + 1$  where  $m_i$  are the exponents of  $\mathfrak{g}$ . I have read in another source that each such generator is termed a **fundamental invariant**<sub>(defined)</sub>. A **Casimir invariant**<sub>(defined)</sub> is a fundamental invariant other than the trivial one. Thus, in this jargon, the ring of invariant polynomials is the polynomial ring over the fundamental invariants.

We now make a definition:

**Definition.** The symbol  $\tau$  (give a name as well?) denotes the set of elements whose orbits are of codimension  $l$  (where  $l$  is the rank of the Lie algebra).

There is a **semisimple plus nilpotent decomposition** <sub>(result assumed)</sub> for a complex reductive Lie algebra: any  $x \in \mathfrak{g}$  can be written uniquely as  $y + z$  satisfying the conditions:

- $y$  is semisimple.
- $z$  is nilpotent.
- $[y, z] = 0$ .

**Claim.** An element  $x \in \mathfrak{g}$  is in  $\tau$  if and only if its nilpotent part  $z$  is a principal nilpotent in  $\mathfrak{g}^y$ , the centralizer of the semisimple part  $y$ .

**Claim.** The space  $H$  of harmonic polynomials coincides with the space of functions spanned by powers of nilpotent linear functionals.

I haven't yet understood this part very well.

5.4. **Bring in the universal enveloping algebra.**

6.

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