DIFFERENTIAL GEOMETRY: MY EVOLUTION IN THE SUBJECT

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ABSTRACT. The subject of differential geometry had interested me a lot while I was in school. Recently, this interest has started getting revived. I have been learning differential geometry formally and informally, and would like to continue to learn more. While I do not think that differential geometry will be my area of specialization, I definitely hope to expand my knowledge in the subject and use it wherever possible.

1. In school

1.1. Differential equations and solution curves. *Time period : January – April 2003*

The high school syllabus talked of the solution of differential equations in terms of "integral curves". There was also a related notion of "field lines" in physics.

I had been interested in "equations" in general. I particularly liked the "degree of freedom" concept for differential equations and wanted a proper framework for understanding these ideas.

1.2. Functional equations. *Time period: March – April 2003*

In preparation for the **International Mathematical Olympiad Training Camp**(camp name)[1], I was sent the book *Functional Equations*(book name) by Dr. Venkatachala, one of India's main Olympiad trainers. On reading this book, I realized that for a general functional equation, it is very difficult to characterize how many degrees of freedom it imposes. I also realized that differential equations are very special functional equations in the sense that they describe the relation between a point and the value of the function at *that point* only. Thus, the "degree of freedom" concept applies well to differential equations.

1.3. Exact form. *Time period: July – December 2003*

During my school days, I also started reading a little vector calculus, using an old book of my grandmother's. This helped me put the results of electrodynamics in perspective. Though I did not see the differential form of the laws of electrodynamics in any high school text, I did come up with all those things myself from the integral forms, after reading some vector calculus. Later, I found that my formulations were the ones used in standard college level electrodynamics books.

One interesting concept was that of "exact form". The notion of "exact differential equation" had been taught as a method of solving differential equations, but the abstract notion of exact form seemed appealing in a different sense. I was also fascinated by the fact that exactness is characterized by a certain differential being zero, which, I was to later learn, is the result that "exact forms are the same as closed forms". I saw many variations of this, in terms of "Green's theorem", "Stokes' theorem" and so on.

1.4. Singer and Thorpe. *Time period : October 2003 – March 2004*

Early on, while in school, I found a book called *Lecture Notes in Elementary Topology and Differential Geometry*(book name) by I.M. Singer and J.A. Thorpe. Towards the end of twelfth standard, I thoroughly studied the first few chapters, which were on topology. But I also had a look at the chapters on Differential Geometry. I didn't understand them much at the time, though.

1.5. Personal exploration. *Time period: 2003 – 04*

Olympiad geometry fascinated me at the time. During the first International Mathematical Olympiad Training Camp, I came across a plethora of results regarding loci based on distances and angles. These results used elementary methods to show that the loci were lines, circles, or other simple curves. Further, as the parameter varied, one got a different curve, and this "family of curves" covered the whole plane. For instance, families of coaxial circles, families of concentric circles, families of confocal conics.

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[1] Learn more at `http://en.wikipedia.org/wiki/IMOTC`

I wanted tools to study the symmetries of these families of curves, both collectively and individually. In a text on differential calculus, I discovered the concepts of "isogonal families" and specifically "orthogonal families". I explored these notions over the families of curves I had encountered.

Later, I also read about plane curves and their analysis using radius of curvature, tangent, normal, and other parameters. I learnt about involute and evolute of a curve, and the geometric interpretations of some results felt nice to me.

2. In CMI

2.1. **A look at the book of Thorpe.** *Time period: October - November 2004*

I located a book by Thorpe (of the same Singer-Thorpe fame) exclusively on Differential Geometry. The book was called *Elementary Topics in Differential Geometry*(book name). For the first time, I was exposed to the *mathematical* definitions of geodesic, meridians, longitudes. I realized that many terms from "geography" and "relativity" had mathematical meaning and mathematical generality.

2.2. **Global calculus course.** *Time period: August - November 2005*

During my third semester, I credited a graduate-level course under Professor Ramanan on **Global Calculus**(course name). I already knew Professor Ramanan, having attended his **Algebra II**(course name) course. I had also seen parts of his *Global Calculus*(book name) book, which formed the basis for his course.

The global calculus course was done in a fairly general manner, and we did not get down to too many nitty-gritty computations. But from talking with Aprameyan (an M.Sc. student), I learnt that much of the material covered was closely related to differential geometry, where the computational aspects were more in focus.

For instance, Professor Ramanan viewed vector fields as the sheaf-theoretic module of derivation over the algebra of functions.

Professor Ramanan also discussed concepts of connections and the connection algebra.

In parallel with Professor Ramanan's regular lectures on global calculus, he was also giving lectures in the CMI-IMSc Representation Theory Seminar. These lectures covered topics like graded-commutative algebras and the representations of Lie algebras. Though I did not get the overall picture, I did understand a lot of tidbits, and these also tied in with the course contents.

Global Calculus is about creating a language by which we can describe, intrinsically and globally, phenomena on a manifold. Aprameyan told me that this language directly helped in differential geometry. Aprameyan had learnt differential geometry from Dr. C.S. Aravinda, so I decided to attend Dr. Aravinda's course the next year.

2.3. **Differential geometry letures at TIFR.** *Time period: June - July 2006*

As part of the **Visiting Students' Research Programme**(camp name), I attended a series of three lectures by Professor Indranil Biswas. The lectures revived a lot of the geometric concepts I had encountered as hobby reading at the end of twelfth standard – particularly notions like radius of curvature, torsion, Serret-Frenet frames and so on.

2.4. **Elementary differential geometry course.** *Time period: August - November 2006*

In my fifth semester, I audited a course on **Elementary Differential Geometry**(course name) by Dr. C.S. Aravinda. Dr. Aravinda discussed a few results about curves, one of which had been dsicussed by Professor Indraneel Biswas.

Dr. Aravinda began by giving the Cartan structural equations in $\mathbb{R}^3$, and the Serret-Frenet equations for curves. He then began on the theory of surfaces, with the ultimate aim being to give a collection of structural equations for surfaces.

He first considered regular surfaces in $\mathbb{R}^3$, and in that setting, obtained notions of principal curvatures, the Gaussian curvature and the mean curvature. He then sought to relate the Gaussian curvature with the inner product defined on the surface, and obtained certain structural equations for the surface.

He then propounded a general notion of "geometric surface", which is a surface with an inner product structure at every point (equivalently, it is a two-dimensional Riemannian manifold). He proved that the Gaussian curvature is an invariant of a geometric surface.

He concluded with the proof of the Gauss-Bonnet theorem, and has also given a flavour of the classification of surfaces.

I also gave a Student Seminar on the Whitney Embedding Theorem. Giving the seminar, I realized the interplay between linear algebra, measure theory, and multivariable calculus.

Course notes and other details can be found on the page:
`http://www.cmi.ac.in/Courses/Aug2006/EDG/`
My own student seminar is available at:
`http://www.cmi.ac.in/~vipul/writeupsandpresentations/proofofwhitneyembedding.pdf`

2.5. **A seminar on the Poincare conjecture.** *Time period: November 2006*

Dr. Aravinda gave a seminar on Perelman's recent proof of the Poincare conjecture at the **Institute of Mathematical Sciences**(place name). This lecture gave me a clear understanding of the statements of Poincare conjecture, the elliptization conjecture and the geometrization conjecture. I also understood the broad motivation behind using Ricci flows to "continuously transform the manifold" to a standard one.

The seminar re-inforced the fact that while intuition plays a very important role in differential geometry, a lot of hardwork and persistence is required to convert the intuition into a concrete and formal proof. Many of the hardest proofs in differential geometry are about showing that certain maps are continuous or differentiable, or about explicitly exhibiting certain maps.

2.6. **Abelian varieties and Global Calculus.** Unfortunately, I have not explored the realm of differential geometry much. However, during the global calculus course, I did spend a lot of time trying to understand various points raised by Professor Ramanan.

Recently, in an **Abelian varieties**(course name) course being taught by Professor Ramanan, concepts of cohomology and complexes were reviewed, and I also learnt about Chern classes. This revived my interest in global calculus.

I took private classes from Professor Ramanan on the Riemann-Roch theorem, its meaning and its implications. These private classes helped me tie in stuff I had picked from diverse sources.

What really fascinate me about all this was the interplay between real differential geometry, complex analysis, advanced algebra, algebraic geometry, and algebraic topology.

2.7. **Riemannian geometry.** *Time period: January - April 2007*

Dr. Aravinda strongly recommended those taking his differential geometry course, take a course in Riemannian geometry that he planned to offer next semester. Unfortunately, due to various other commitments, Dr. Aravinda was unable to teach the Riemannian geometry course. Luckily, however, the course was not scrapped and Dr. M. K. Vemuri is teaching it instead.

Despite many scheduling problems and cancellation of many scheduled lectures, Dr. Vemuri covered a lot of ground in the Riemannian geometry course. I particularly enjoyed the various curvature tensors, the deep results about their tensoriality, and I liked the geometric proof that Vemuri gave for the foundational Hopf-Rinow thoerem.

Towards the end of the course, Dr. Vemuri also taught a bit about geodesics. I plan to study more on geodesics and geodesic variations in greater detail.

3. Ricci flows

Time period: March – April 2007

On the recommendation of Dr. C. S. Aravinda, I started reading some of Hamilton's papers on the Ricci flow. Unfortunately, I found these papers very hard to read, and for this and various other reasons, the reading project got suspended.

Some time in the end of March (or beginning of April) the CMI library got two new books, one titled *An Introduction to the Ricci Flow*(book name) by Chow and Kampf, and the other a lecture series by Topping. I found the book by Chow and Kampf nice reading, and started documenting whatever I learnt from there, along with various other documentation I was doing on Riemannian geometry.

3.1. **A differential geometry wiki.** More as an aid to private learning rather than for public use, I started documenting whatever I was learning in differential geometry on a wiki:
`http://diffgeom.wiki-site.com`

3.2. **The next step: computational aspects.** In the cmoing two months, I plan to spend some time focussing on the computational aspects of differential geometry, namely: given a description of a curve, surface, or manifold, and a point on it, how can one compute the various curvatures and curvature tensors at the point (in suitable local coordinates)? I am currently reading texts on differential geometry with Mathematica.

4. For the future

4.1. **As a side-interest.** Since my primary area of focus is algebra, I do not think I will be pursuing research within differential geometry. Nonetheless, I think a good understanding of differential geometry will help me understand Lie groups, where a "purely" algebraic approach is very difficult to manage. This, in turn, will be useful in representation theory, which is one of my main areas of interest at the moment.

The close connections between algebraic topology and differential geometry are intriguing, and thus it may happen that a better understanding of differential geometry is helpful for algebraic topology.

4.2. **Understanding physics.** A lot of physics is done over manifolds these days. I have some friends who are majoring in physics, and they have told me of the significance of manifolds in general relativity. I plan to learn more about general relativity as well as quantum mechanics, and I believe that a sound base in differential geometry will help me.

Here is a list of books on differential geometry:

(1) *Lecture notes on Elementary Topology and Differential Geometry*(book name) by I. M. Singer and Jon A. Thorpe.

How I used this book: I first used this book for point-set topology. I have not studied differential geometry systematically from it. However, I got an introduction to the general motivations and ideas of differential geometry from this book. I believe it is an excellent book for reference. It is very terse and to-the-point, but the explanations are easy to understand for a person who can grasp things from basic definitions and a few examples.

(2) *Elementary Topics in Differential Geometry*(book name) by Thorpe.

How I used the book: I studied this book for about 1-2 weeks. It familiarized me with notions of local coordinate charts, Gauss maps, vector fields, geodesics, meridians and so on.

(3) *An Introduction to the Ricci Flow*(book name) by Chow and Kampf.

How I used this book: I studied parts of this book to understand some aspects to the Ricci flow

(4) *Riemannian geometry*(book name) by Jurgen Jost

Other authors whose books I have heard of (but never read/studied firsthand) include: Spivak, do Carmo, Barrett O'Neill.

For course notes of the course on Differential Geometry that I am attending: `http://www.cmi.ac.in/Courses/Aug2006/EDG/`

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