ALGEBRAIC GEOMETRY: MY DEVELOPMENT IN THE SUBJECT

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Abstract. Algebraic geometry has been a buzzword in CMI, IMSc as well as TIFR, so I have naturally developed an interest in the subject. Here, I describe how my interest evolved, and what I plan to do to make both my level of knowledge and my level of interest grow.

1. In school times

1.1. **Coordinate geometry.** *Time period: April 2002 – March 2004*

Algebraic geometry is not simply an advanced version of the coordinate geometry we learn in school. Nonetheless, many of the concepts I learnt in coordinate geometry sowed the seeds of exploration which led me to appreciate the subject of algebraic geometry better when I saw it.

One of these was the concept of "coaxial circles" and "radical axis". Given the equations of two circles, say S_1 and S_2, the system of coaxial circles is the collection of all circles whose equations can be obtained via linear combinations of these. I saw this similar "linear combination" idea being used in many geometry problems to parameterize families. This led me to raise a few questions:

- Given any two functions, all functions obtained as linear combinations, or even as combinations with polynomial coefficients of these, will vanish on the set of intersections. Is it true that those are the only functions?
- Obviously that isn't true. Then I thought of this: given a set of points, what is the collection of functions that vanish on that point. Are there some basic functions?

I don't think I viewed these questions as clearly as I can in hindsight, but I had these questions vaguely in my mind.

1.2. **Hit upon Dummit and Foote.** *Time period: May - August 2004*

During the **Kishore Vaigyanik Protsahan Yojana**(camp name) camp I bought the book *Abstract Algebra*(book name) by Dummit and Foote. I read the section in the book on Hilbert's nullstellensatz. I did not go through the proof completely but I did have a careful look at the theorem and a few days after I read it, it clicked to me that this was the theorem that answered the questions that I had raised. Namely, it gave an exact correspondence between certain kinds of sets called closed sets and certain collections of functions called radical ideals.

Two years later, I was to give a talk on this topic as part of the Student Talks initiative. Talk slides are available at:

`http://www.cmi.ac.in/~vipul/studenttalks/coaxialcirclestrianglecentersandthenullstellensatz.pdf`

2. In college now

2.1. **Not much progress to begin with.** *Time period : August - November 2004*

I was getting exposed to many new subjects in college and algebraic geometry seemed to require a number of prerequisites, so I did not devote much attention to the subject during the first semester. However, I did get to understand the leading motivations behind the subject.

Professor Balaji suggested that I first familiarize myself with commutative algebra and then plunge into algebraic geometry. He recommended. the book **Commutative Algebra**(course name) by Atiyah and Macdonald. I could not procure a copy for myself but I had a look at it and did a few exercises.

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2.2. **A talk with Professor Ramanan.** *time period: February-March 2004*

Our **Algebra II**(course name) teacher, Professor Ramanan, was a versatile algebraic geometer. He was at the time giving a reading course to an M.Sc. student in algebraic geometry. I had a look at the lecture notes which clarified things in more ring-theoretic terms. I also asked him what was there in algebraic geometry beyond the Nullstellensatz.

Professor Ramanan explained that algebraic geometry could be done over various fields, whereas topological notions could be discussed only over fields like $\mathbb{R}$ or $\mathbb{C}$, or (as I add with my new experience) over the p-adic completions of $\mathbb{Q}$. Finite fields, on the other hand, are fairly devoid of topology, and function fields over $\mathbb{Q}$ or over $\mathbb{C}$ also are not very amenable to topological study. So we try to convert the *topological* concepts into purely *algebraic* concepts which work over any field.

Professor Ramanan also converted the statements of Hilbert's nullstellensatz to the statement that the polynomial ring over a field is a Hilbertian ring, also called Hilbert ring or Jacobson ring. This led me further along the path of thinking of results in terms of properties. I explored and discovered that the ring $C[0,1]$ is not Hilbertian.

2.3. **Introduction to sheaves.** *Time period: August-September 2005*

In Professor Ramanan's **Global Calculus**(course name) course, I learned about sheaves and their use to study functions. The course focussed on the use of sheaves for differentiable functions, but based on brief allusions by Professor Ramanan and some reading of my own, I figured out the use in algebraic geometry as well.

Later, I attended a few talks and got a clear idea of what "variety" and "scheme" means.

2.4. **Commutative algebra towards algebraic geometry.** *Time period: August-December 2005*

Some time in the beginning of my second year, I became very interested in ring theory and commutative algebra. I had a look at two books: *Commutative algebra*(book name) by Atiyah and Macdonald, and *Commutative algebra with a view towards algebraic geometry*(book name) by David Eisenbud.

These books showed me that algebraic geometry could be done over an arbitrary Noetherian ring, and also demonstrated what additional ring-theoretic assumptions were necessary to make the algebraic geometry a "nice" one. Later, as I became more interested, I started reading the book *Algebraic Geometry*(book name) by Hartshorne.

2.5. **Personal classes with Professor Ramanan.** *Time period: 2005 - 06*

During the third and fourth semesters and for two weeks after the fourth semester, I took personal classes under Professor Ramanan. In one class, he explained the distinction between variety and scheme. He later explained primary decomposition from the algebraic geometric perspective.

After the fourth semester, Professor Ramanan explained some fine points of complex analysis, such as divisors, Riemann-Roch thoerems, and reciprocity. He again alluded to related ideas in algebraic geometry.

2.6. **Topology towards algebraic geometry.** *Time period: January – April 2006*

In the **Topology**(course name) course, Professor Balaji did some category theory, and also covered basic algebraic topology from an algebraic geometry perspective. He gave a definition of "simply connected covering space" in terms of trivializability of all covering maps. He said that these ideas were useful for generalizing to other scenarios. This fuelled my interest in algebraic geometry, although I did not pursue the interest further at the time.

2.7. **The paper under Professor Dipendra Prasad.** *Time period : June - July 2006*

When applying for the **Visiting Students Research Programme**(camp name) (VSRP) at **Tata Institute of Fundamental Research**(place name) I specified algebraic geometry as one of my areas of interest. Professor Dipendra Prasad, the VSRP Coordinator, offered to be my guide and gave me a paper on "Lie Group Representations of Polynomial Rings" written by Bertram Kostant.

I also had interactions with Professor Dipendra Prasad where I understood the role and relation between algebraic geometry and Lie groups.

The paper was my first systematic introduction to all the subtleties and complexities of invariant theory and the relation between ideal, vanishing sets and groups of automorphisms.

The paper was also my first experience at studying and mastering a portion of an intricate research-level paper. Earlier, I used to only read small segments of the paper to get a general idea, and study a few salient proofs. Now, I acquired the discipline to read a paper in full detail.

I have documented some of my personal experiences with reading the paper at:
`http://whatisresearch.blogspot.com`

2.8. **Lecture series in TIFR.** *Time period: June - July 2006*

These lecture series led me to think along new lines. One was under Professor M.S. Raghunathan. I saw the importance of results from differential geometry, such as the implicit function theorem and Sard's theorem, in the context of Lie groups and algebraic geometry.

2.9. **Course in Abelian varieties.** *Time period: August - November 2006*

In my fifth semester, I credited a course in **Abelian varieties**(course name) being offered by Professor Ramanan. Although a study of Abelian varieties usually invokes heavy prerequisites of algebraic geometry, Professor Ramanan has chosen a bare-bones approach that uses a minimum of background. So far, I have understood and enjoyed the contents of the course.

Professor Ramanan mentioned the *Geometry Algebraique Geometry Analytique* or GAGA philosophy as part of his course. The basic statement is to establish a correspondence analytic properties of a complex-analytic manifolds with its algebraic properties when viewed as a complex projective variety. I plan to study the work done in this regard later.

The mid-semester and end-semester papers for this course can be found at:
`http://www.cmi.ac.in/~vipul/courses/abelianvarieties`
I also plan to put up a course summary and some lecture notes on that page soon.

3. Current work

3.1. **Papers on invariant theory.** Currently, I am working on Weitzenbock's theorem and related results, based on a small paper by Seshadri. I plan to put up details of what I have worked on soon.

Trying to understand these ideas has also led me to revise and strengthen my knowledge of results from commutative algebra.

3.2. **Revisiting Kostant's paper.** This semester I am studying a course titled **Lie-theoretic methods in analysis**(course name). In this course, I studied some notions about Lie algebras that led me to revisit certain portions of Kostant's paper. I am also independently reading the representation theory of Lie algebras, and trying to understand the relationship with algebraic geometry.

Books in algebraic geometry and commutative algebra:

(1) *Algebraic Geometry*(book name) by Hartshorne
(2) *Commutative Algebra with a view towards algebraic geometry*(book name) by Dvid Eisenbud
(3) *Commutative Algebra*(book name) by Atiyah and Macdonald