10/4/2014

NON-CONVEX ABSTRACTIONS

There are two main zone manipulations involved in the reachability algorithm:

- 1. Successor computation
- 2. Inclusion checking: $Z \subseteq Closure_M(Z')$

We have already seen how to compute successors in $O(n^2)$ time where n is the number of clocks. In this lecture, we will see how to perform the test $Z \subseteq Closure_M(Z')$ in $O(n^2)$ time too.

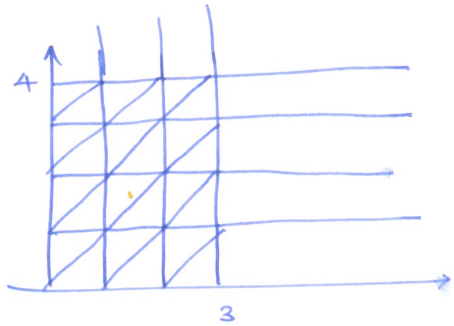
1. What is $closure_M(Z)$?

M is a "bounds function" that assigns a value to every clock: $M : X \longmapsto \mathbb{N} \cup \{-\infty\}$

~~A clock x is assigned~~

For instance, $M_1(x) = 5$, $M_1(y) = 2$, $M_1(z) = -\infty$ is a valid bounds function.

Given M, recall that $\sim_M$ stands for region equivalence, whose equivalence classes are called REGIONS. For instance, over 2 clocks, if $M(x) = 3$, $M(y) = 2$, the classes look like below:



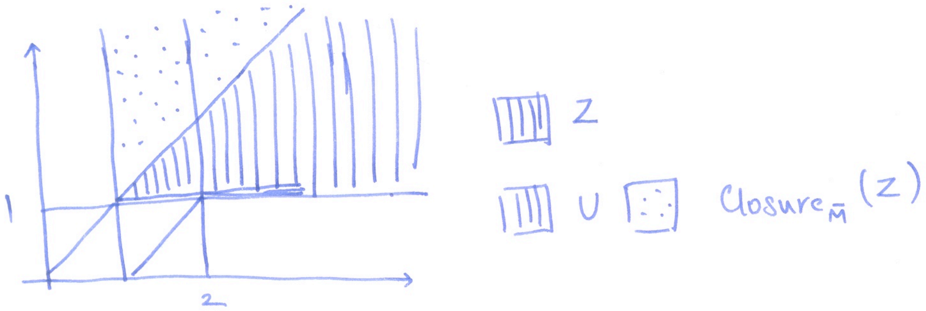
For a valuation v, we denote $[v]_M$ to be the equivalence class of valuation v. this is exactly one region.

~~$Closure_M(Z) = \{ \bigcup \{ v_M$~~

~~$Closure_M(Z) = \bigcup \{ [v]_M \mid v \in Z \}$~~

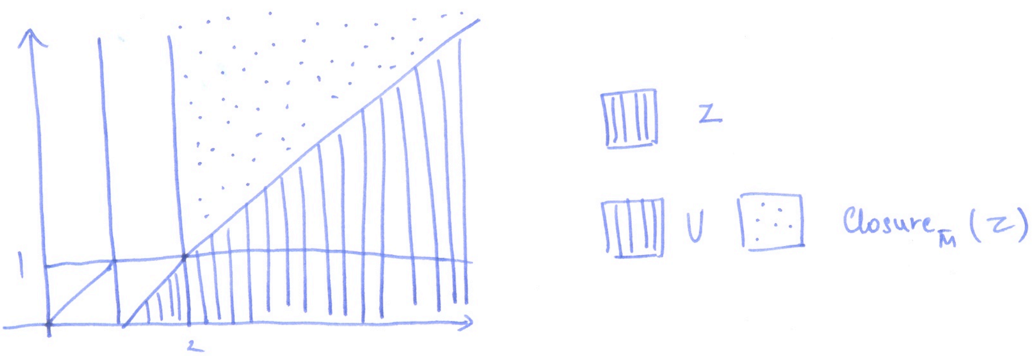
$$Closure_M(Z) = \bigcup_{v \in Z} [v]_M$$

$Closure_M$ of a zone is just the union of regions intersecting Z.



$\bar{M}(x) = 2$
$\bar{M}(y) = 1$

Note that $Closure_M$ of a zone need not have to be zone. It could be a non-convex set, as illustrated below:



$\bar{M}(x) = 2$
$\bar{M}(y) = 1$

2. Efficient inclusion $Z \subseteq \mathrm{Closure}(Z')$:

To get to an efficient algorithm for $Z \subseteq \mathrm{Closure}_M(Z')$, we will make use of the following steps:

Proposition 1: $Z \nsubseteq \mathrm{Closure}_M(Z')$ iff there exists $v \in Z$ s.t. $[v]_M \cap Z'$ is empty.

Proposition 2: Let R, Z' be a region (over M) and a non-empty zone respectively. Then $R \cap Z'$ is empty iff there exist variables x, y s.t.

$$Z'_{yx} + R_{xy} < (\leq, 0)$$

Here Z'_{yx} denotes the weight of $y \longrightarrow x$ in the canonical distance graph representing Z'. Similarly R_{xy} denotes the weight of $x \longrightarrow y$ in the canonical distance graph representing R.

Theorem 3: Let Z, Z' be non-empty zones. Then $Z \nsubseteq \mathrm{closure}_M(Z')$ iff there exist variables x, y s.t.

$$Z_{x0} \geq (\leq, -M_x) \text{ and } Z'_{xy} < Z_{xy} \text{ and } Z'_{xy} + (<, -M_y) < Z_{x0}$$

~~Note that the above comp~~

We will now provide proof of Proposition 2. The proof of Proposition 1 is direct from definition of $Closure_M$. Proof of the final theorem uses Proposition 2. But the steps are very technical and not needed as part of the course.

Let us now look at Proposition 2. It says that ~~a~~ ~~if~~ R and Z' don't intersect iff there are 2 variables s.t. projection of R and Z' don't intersect. Such a theorem need not be true in general for any two objects. Regions and zones are special sets. We have a lot of control over their constraints describing them and hence we get such a theorem.

To prove proposition 2, we will need the foll. ~~Lemma~~ property in Lemma 5 that exploits the special structure of regions.

~~Lemma 4~~

A variable x is said to be bounded in a region R if $R_{0x} \leqslant M_x$. Recall that R_{0x} denotes weight of $0 \to x$ in canonical distance graph of R.

Lemma 4: Let R, Z' be ~~none~~ region and non-empty zone. ~~Consider~~ Let G_R and $G_{Z'}$ be their canonical distance graphs. Consider $\min(G_R, G_{Z'})$. $R \cap Z'$ is empty iff $\min(G_R, G_{Z'})$ has a negative cycle.

$\longrightarrow$ proved in an earlier notes.

Lemma 5: Let x, y be bounded variables of R appearing in some negative cycle N of $\min(G_R, G_{Z'})$. Let the edge weights be $x \xrightarrow{<_{xy}\, c_{xy}} y$ and $y \xrightarrow{<_{yx}\, c_{yx}} x$ in G_R. If the value of the path $x \to \dots \to y$ in N is strictly less than $(<_{xy}, c_{xy})$, then $x \to \dots \to y \xrightarrow{<_{yx}\, c_{yx}} x$ is a negative cycle.

Proof:

Let the path $x \to \dots \to y$ in N have weight $(<, c)$. Now since x and y are bounded variables of R, we can have either $y - x = d$ or $d-1 < y - x < d$ for some integer d.

In the first case, we have the foll edges in G_R:

$$\begin{tikzcd} x \arrow[r, bend left, "\leq d"] & y \arrow[l, bend left, "\leq -d"] \end{tikzcd}$$

That is: $(<_{xy}\, c_{xy}) = (\leq d)$ and $(\leq_{yx}\, c_{yx}) = (\leq -d)$. Since by hypothesis, $(<, c)$ is strictly less than $(\leq, d)$, we have that either $c < d$ or $c = d$ and $< \, = \, <$. Hence

$$(<, c) + (\leq, d) < (\leq, 0)$$

showing that $x \to \dots \longrightarrow y \xrightarrow{<_{yx}\, c_{yx}} x$ is a negative cycle.

In the second case, we have the foll edges in G_R:

$$\begin{tikzcd} x \arrow[r, bend left, "<d"] & y \arrow[l, bend left, "<-d+1"] \end{tikzcd}$$

that is: $(<_{xy}\; c_{xy}) = (<\, d)$ and $(<_{yx}\; c_{yx}) = (<, -d+1)$

Here as $(<, c) < (<_{xy}, c_{xy})$, we need to have $c < d$.

Hence $(<, c) + (<, -d+1) < (\le, 0)$

and hence $x \to \cdots y \xrightarrow{<_{yx}\, c_{yx}} x$ is a negative cycle.

$\square$

Proof of Proposition 2:

$\Leftarrow$: Suppose $\exists\, x, y$ s.t. $Z'_{yx} + R_{xy} < (\le, 0)$. This means there is a negative cycle in $\min(G_R, G_{Z'})$. Hence ~~G~~ $Z' \cap R = \phi$.

$\Rightarrow$: this is the tougher part. Suppose $R \cap Z' = \phi$. There exists a $-$ive cycle N in $\min(G_R, G_{Z'})$. We want to show that ~~any~~ ~~a~~ negative cycle N can be reduced to the form:

$$x \xrightarrow{Z'_{xy}} y \xrightarrow{R_{yx}} x.$$

As G_R and $G_{Z'}$ are canonical, we can assume that no two consecutive edges in N come from same graph.

$$N: \quad x_1 \xrightarrow{Z'} x_2 \xrightarrow{R} \xrightarrow{Z} \xrightarrow{R} \cdots \xrightarrow{R} x_1$$

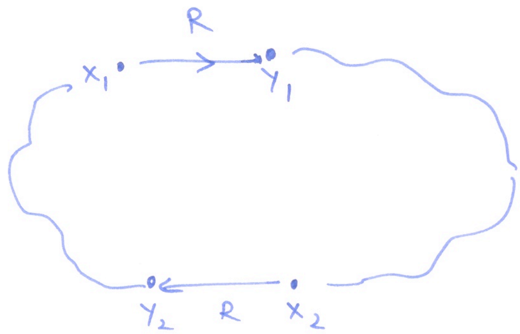
Also all the ~~values~~ weights in N should be finite.

Note that if $u \xrightarrow{R} w$ is a finite value in G_R, then <u>w has to be a bounded variable</u> in R.

Take 2 edges that come from R in N.

$X_1 \xrightarrow{R} Y_1 \qquad\qquad X_2 \xrightarrow{R} Y_2$

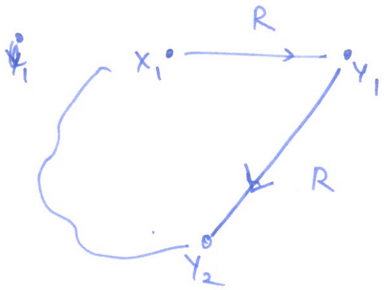
~~X_2~~ Y_1 and Y_2 are bounded. ~~Hence by Lemma 5~~,



By Lemma 5, if $Y_1 \to \dots \to Y_2$ is smaller than $Y_1 \xrightarrow{R} Y_2$ then we get a smaller negative cycle in which the edge $X_1 \xrightarrow{R} Y_1$ is not present.

Otherwise, if $Y_1 \longrightarrow \dots \to Y_2$ is bigger than $Y_1 \xrightarrow{R} Y_2$

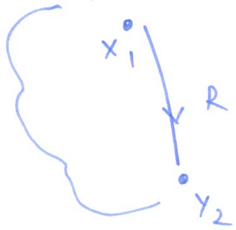
then



is a −ive cycle.

~~As R is~~ can

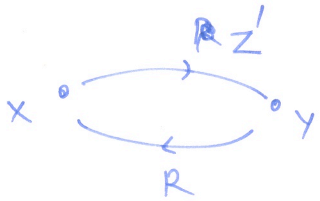
But as R is canonical,



is a negative cycle.

Here we have reduced 2 R edges to a single R edge.

As 2 R edges reduce to single R edge in both cases, we can reduce N to:



containing only one R edge.

$\square$

<u>Final remark</u>: Theorem 3 gives the final inclusion test.

Check that the complexity is $O(n^2)$.