The Separation Problem: An Introduction and a Transfer Theorem

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Joint work with Thomas Place

ACTS 2015, Chennai — February 9, 2015

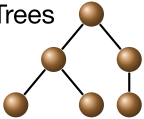
Objects we consider

Structures

Words

ababcbaa

Trees



Descriptive Formalism

First-Order Logic (**FO**)

Piecewise Testable ($\mathcal{B}\Sigma_1$)

2-Variables **FO** (**FO**$_2$)

Fragments Σ_i, $\mathcal{B}\Sigma_i$

Locally Threshold Testable (**LTT**)

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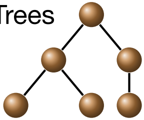

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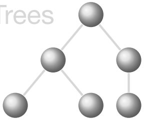

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For this talk

First-order logic on words

First-order logic, with only the linear order '$<$'.

$$a \;\; b \;\; b \;\; b \;\; c \;\; a \;\; a \;\; a \;\; c \;\; a$$

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$$\begin{array}{cccccccccc} a & b & b & b & c & a & a & a & c & a \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

- A word is as a sequence of labeled positions that can be quantified.
- Unary predicates $a(x), b(x), c(x), \ldots$ testing the label of a position.
- One binary predicate: the linear-order $x < y$.

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Example: every a comes after some b

$$\forall x\ a(x) \Rightarrow \exists y\ (b(y) \wedge (y < x))$$

Why look at fragments in addition to full FO?

- Simple formulas are better (aesthetically, algorithmically).
- Some parameters making formulas complex:
  - Number of quantifier alternations,
  - Allowed predicates,
  - Number of variable names.

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Membership Problem for a fragment $\mathcal{F}$

- **INPUT** A language L.
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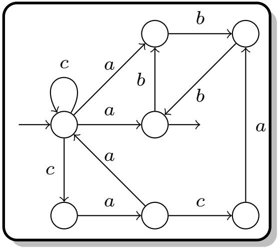

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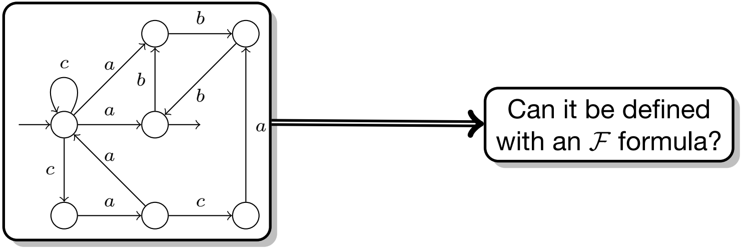

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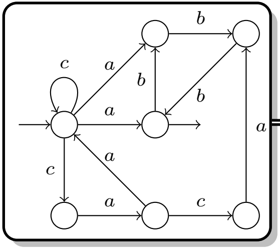


Can it be defined with an $\mathcal{F}$ formula?

Schützenberger'65, McNaughton and Papert'71

For L a regular language, the following are equivalent:

- L is **FO**-definable.
- The syntactic monoid of L satisfies $u^{\omega+1} = u^{\omega}$.

Fragments of FO

- A fragment is obtained by restricting
  - Number of quantifier alternations,
  - Allowed predicates,
  - Number of variable names.

- FO$(<)$, FO$(<, +1)$ and FO$(<, +1, min, max)$: same expressiveness.

$\Rightarrow$ Allowing '$=$' but not '$<$' yields distinct fragments.

$$\Sigma_1(<), \quad \Sigma_1(<, +1), \text{ and } \Sigma_1(<, +1, min, max)$$

- We do not want to prove membership multiple times.

Some well-known fragments

Weak variant	Strong variant
FO$(=)$	FO$(=, +1)$
FO$^2(<)$	FO$^2(<, +1)$
$\Sigma_n(<)$	$\Sigma_n(<, +1, min, max)$
$\mathcal{B}\Sigma_n(<)$	$\mathcal{B}\Sigma_n(<, +1, min, max)$

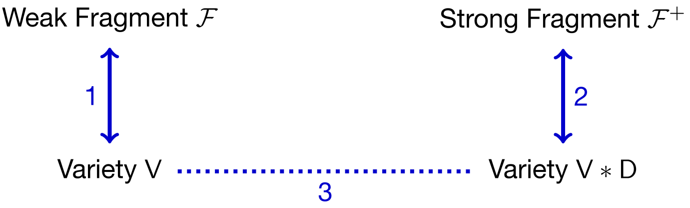
- Problem: Solve membership for strong variants without reproving everything nor mimicking the proof.

A generic result for membership

- Problem Solve membership for strong variants without reproving everything nor mimicking the proof.
- S. Eilenberg Each fragment is associated the class of finite monoids recognizing a language from the fragment.
  Example: FO $\longleftrightarrow [x^\omega = x^{\omega+1}]$.
- H. Straubing 1985 + M. Kulfleitner & A. Lauser 2014: generic result.

Weak Fragment $\mathcal{F}$	Strong Fragment $\mathcal{F}^+$
$\updownarrow$	$\updownarrow$
Variety V	Variety V $*$ D

Straubing's Theorem



1. Show the correspondence between $\mathcal{F}$ and algebraic variety V.
2. In most cases, the enriched fragment $\mathcal{F}^+$ corresponds to V $*$ D.
3. In most cases, V $\mapsto$ V $*$ D preserves decidability.

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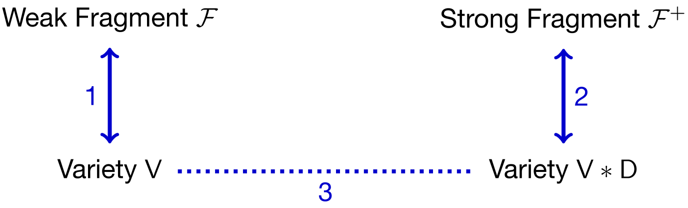

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2. In most cases, the enriched fragment $\mathcal{F}^+$ corresponds to V $*$ D.
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Remarks

- One need to establish the correspondence 1.
- That V $\mapsto$ V $*$ D preserves decidability is a difficult result.

An alternative approach

B. Steinberg 2001

- All fragments share a property entailing decidability of membership.
- This property is preserved through enrichment.

Even if we are interested in the membership problem for $\mathcal{F}$,
it does not give sufficient information to reason about $\mathcal{F}$.

Why we want more than membership

If the membership answer for L

- is **YES**
  - All “subparts” of the minimal automaton of L are $\mathcal{F}$-definable.
- is **NO**, then even if $\mathcal{F}$ can talk about L:
  - We have little information.
  - Eg, defining L in FO would require differentiating some u^{ω} and $u^{\omega+1}$.

Motivations for Separation

- Need more general techniques to extract information for all languages.
- Cannot start from canonical object for the separator, which is unknown.
- Therefore, may give insight to solve harder problems.

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- 2 examples of “transfer results”:
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  - decidability of separation for level Σ_i of the quantifier alternation hierarchy entails decidability of membership for Σ_{i+1}.

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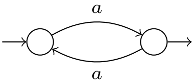
$\Rightarrow$ We shouldn’t restrict ourselves to membership

Beyond membership: Separation

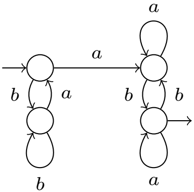
Decide the following problem:

Take two regular languages L_1, L_2

L_1



L_2



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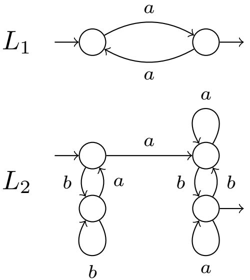
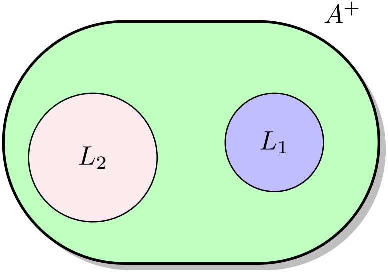


Can L_1 be separated from L_2 with an $\mathcal{F}$ formula?



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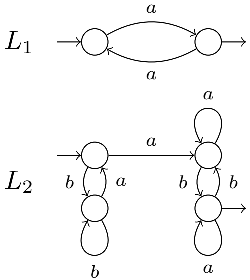

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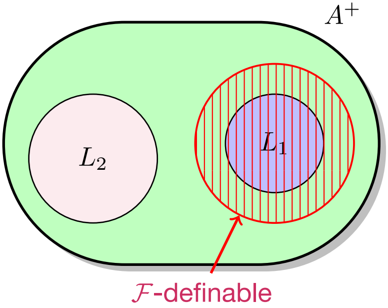

Beyond membership: Separation

Membership can be formally reduced to separation

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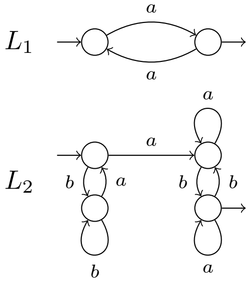

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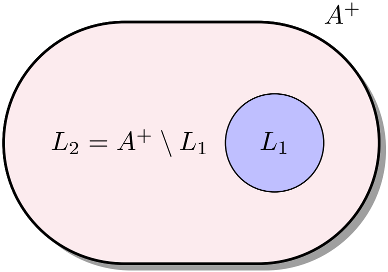

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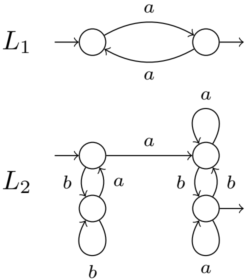

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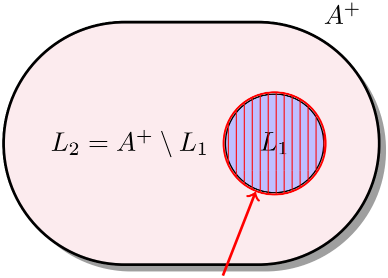

$\mathcal{F}$-separable from complement

$\Leftrightarrow$

$\mathcal{F}$-definable

Related work

- Separation **already considered** in an algebraic framework.
- First result by K. Henckell ’88 for **FO**, then for several natural fragments.
- Purely algebraic proofs, hiding the combinatorial and logical intuitions.
- Transfer result of this talk already obtained by Ben Steinberg ’01.
- Simpler proof techniques.

A toy example: Separation for FO$(=)$

- In FO$(=)$, one can just count occurrences of letters, up to a threshold.
- Example: at least 2 a's: $\exists x, y \;\; x \neq y \wedge a(x) \wedge a(y)$.
- FO$(=)$ can express properties like

  at least 2 a's, no more than 3 b's, exactly 1 c.

- How to decide separation for FO$(=)$?

A toy example: Separation for FO$(=)$

- Let $\pi(u) \in \mathbb{N}^A$ be the commutative (aka. Parikh) image of u.

$$\pi(aabad) = (3, 1, 0, 1).$$

Parikh's Theorem

For L context-free, $\pi(L)$ is (effectively) semilinear.

- For $\vec{x}, \vec{y} \in \mathbb{N}^A$, $\quad \vec{x} =_d \vec{y} \quad$ if $\quad \forall i$: $x_i = y_i$ or both $x_i, y_i \geqslant d$.

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Fact

Languages L_1, L_2 are not FO$(=)$-separable iff

$$\forall d \quad \exists u_1 \in L_1\, \exists u_2 \in L_2, \quad \pi(u_1) =_d \pi(u_2).$$

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$\Leftarrow$ Assume there is an FO$(=)$-separator K, say of threshold d.
Then $L_1 \subseteq K \implies u_1 \in K \implies u_2 \in K$, impossible since $u_2 \in L_2$.

A toy example: Separation for FO($=$)

Fact

Languages L_1, L_2 are not FO($=$)-separable iff

$$\forall d \quad \exists \vec{x}_1 \in \pi(L_1) \, \exists \vec{x}_2 \in \pi(L_2), \quad \vec{x}_1 =_d \vec{x}_2.$$

By Parikh's Theorem, decidability follows from that of Presburger logic.

Separation for FO$(=, +1)$

- FO$(=)$ can just count occurrences of letters up to a threshold.
- FO$(=, +1)$ can just count occurrences of infixes up to a threshold.

  There exist at least 2 occurrences of $abba$
  and the word start with ba.

- For membership, decidability follows from a delay theorem:
  To test FO$(=, +1)$-definability, one can look at infixes of bounded size.

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- Membership proof is not trivial. Transferring separability is easier.

The transfer result

Let $\mathcal{F}$ be one of FO$(=)$, FO$^2(<)$, $\Sigma_n(<)$, $\mathcal{B}\Sigma_n(<)$.

Main result

$\mathcal{F}^+$-separability reduces to $\mathcal{F}$-separability.
For any regular L, one can build a regular language $\mathbb{L}$ such that

$$L_1 \text{ and } L_2 \text{ are } \mathcal{F}^+\text{-separable iff. } \mathbb{L}_1 \text{ and } \mathbb{L}_2 \text{ are } \mathcal{F}\text{-separable.}$$

- Simple.
- Extends to infinite words.
- Mostly generic and Constructive

from an $\mathcal{F}$ formula separating $\mathbb{L}_1$ and $\mathbb{L}_2$,
build
an $\mathcal{F}^+$ formula that separates L_1 from L_2.

Well formed words

- Intuition: adding +1 makes it to inspect infixes.
- Use regularity of input languages: large infixes will contain loops.

  Fix $\alpha : A^+ \to S$ recognizing L_1 and L_2.

  $$L_i = \alpha^{-1}(F_i).$$

  $E(S)$ = set of idempotents of S.

  $$E(S) = \{e \in S \mid ee = e\}$$

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- New alphabet

$$\mathbb{A}_\alpha = (E(S) \times S \times E(S)) \ \cup \ (S \times E(S)) \ \cup \ (E(S) \times S) \ \cup \ S.$$

- Well formed word: either a single $s \in S$, or

$$(s_0, f_0) \cdot (e_1, s_1, f_1) \cdots (e_n, s_n, f_n) \cdot (e_{n+1}, s_{n+1})$$

with $f_i = e_{i+1}$.

Extending the morphism on well-formed words

- Well formed word over $\mathbb{A}$: either a single $s \in S$, or

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Fact. The language of well formed words is regular.

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- Morphism $\beta : \mathbb{A}^+ \to S$, defined by

$$\beta(s) = s \qquad \beta(e, s, f) = esf$$
$$\beta(e, s) = es \qquad \beta(s, f) = sf$$

Therefore,

$$\beta[(s_0, e_1) \cdot (e_1, s_1, e_2) \cdot (e_2, s_2, e_3) \cdots (e_n, s_n, e_{n+1}) \cdot (e_{n+1}, s_{n+1})]$$
$$=$$
$$s_0 e_1 s_1 e_2 s_2 e_3 \cdots e_n s_n e_{n+1} e_{n+1} s_{n+1}$$

Associated language of well-formed words

- To a language $L \subseteq A^+$ recognized by α, associate $\mathbb{L} \subseteq \mathbb{A}^+$.

$$\begin{aligned}\mathbb{L} &= \{w \in \mathbb{A}^+ \mid \beta(w) \in \alpha(L)\} \\ &= \beta^{-1}(\alpha(L)).\end{aligned}$$

Fact. The language $\mathbb{L}$ associated to L is (effectively) regular.

Main result again

Let $\mathcal{F}$ be one of FO$(=)$, FO$^2(<)$, $\Sigma_n(<)$, $\mathcal{B}\Sigma_n(<)$. and $\mathcal{F}^+$ be its enrichment.

Let $L_1, L_2 \subseteq A^+$ be regular languages recognized by α.

L_1 and L_2 are $\mathcal{F}^+$-separable iff. $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathcal{F}$-separable.

A consequence for membership

Let $\mathcal{F}$ be one of FO$(=)$, FO$^2(<)$, $\Sigma_n(<)$, $\mathcal{B}\Sigma_n(<)$. and $\mathcal{F}^+$ be its enrichment.

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Main result (separation)

L_1 and L_2 are $\mathcal{F}^+$-separable iff. $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathcal{F}$-separable.

$\Rightarrow$ **Separation** decidable for enrichment of FO$(=)$, FO$^2(<)$, $\mathcal{B}\Sigma_1$, Σ_n $n \leqslant 3$.

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Corollary (membership)

If in addition $\mathcal{F}$ can define the set of well formed words:

L is $\mathcal{F}^+$-definable iff. $\mathbb{L}$ is $\mathcal{F}$-definable.

$\Rightarrow$ **Membership** decidable for $\mathcal{B}\Sigma_2(<, +1)$ and $\Sigma_4(<, +1)$.

Proof of the corollary for membership

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Proof. Let $K = A^+ \setminus L$ and $\mathbb{K}$ associated to K.
$\mathbb{K}$ and $\mathbb{L}$ partition the set of all well-formed words.

($\Longleftarrow$) $\mathbb{L}$ is $\mathcal{F}$-definable $\implies$ $\mathbb{L}$ is $\mathcal{F}$-separable from $\mathbb{K}$

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($\Longleftarrow$) $\mathbb{L}$ is $\mathcal{F}$-definable $\implies$ $\mathbb{L}$ is $\mathcal{F}$-separable from $\mathbb{K}$
$\implies$ L is $\mathcal{F}^+$-separable from K *by Main result*

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Proof. Let $K = A^+ \setminus L$ and $\mathbb{K}$ associated to K.
$\mathbb{K}$ and $\mathbb{L}$ partition the set of all well-formed words.

($\Longleftarrow$) $\mathbb{L}$ is $\mathcal{F}$-definable $\Longrightarrow$ $\mathbb{L}$ is $\mathcal{F}$-separable from $\mathbb{K}$
$\Longrightarrow$ L is $\mathcal{F}^+$-separable from K *by Main result*
$\Longrightarrow$ L is $\mathcal{F}^+$-definable.

Proof of the corollary for membership

Main result (separation)

L_1 and L_2 are $\mathcal{F}^+$-separable iff. $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathcal{F}$-separable.

Corollary (membership)

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($\Longrightarrow$) L is $\mathcal{F}^+$-definable $\Longrightarrow$ L is $\mathcal{F}^+$-separable from K
$\Longrightarrow$ $\mathbb{L}$ is $\mathcal{F}$-separable from $\mathbb{K}$ by $\mathbb{S}$

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$\Longrightarrow$ $\mathbb{L}$ is $\mathcal{F}$-separable from $\mathbb{K}$ by $\mathbb{S}$
$\Longrightarrow$ $\mathbb{L} = \mathbb{S} \cap (\mathbb{L} \cup \mathbb{K})$ is $\mathcal{F}$-definable.

From $\mathcal{F}$-separation to $\mathcal{F}^+$-separation

Main result (separation, generic direction)

If $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathcal{F}$-separable, then L_1 and L_2 are $\mathcal{F}^+$-separable.

Proof. Associate to $w \in A^+$ a word $\lfloor w \rfloor \in \mathbb{A}_\alpha^+$ such that $\alpha(w) = \beta(\lfloor w \rfloor)$.

- u_x: infix of length $|S|$ ending at x.

$$\cdots\ aba\underbrace{aaababb\overset{\overset{x}{\downarrow}}{a}}_{|S|}a\ \cdots$$

- Position x is distinguished if $\exists e \in E(S)$ such that $\alpha(u_x) \cdot e = \alpha(u_x)$.
- $x_1 < \cdots < x_n$ = distinguished positions induce a splitting

$$w = w_1 \cdot w_2 \cdots w_{n+1}$$

- Define $\lfloor w \rfloor \in \mathbb{A}_\alpha^+$ by choosing e_i canonically and

$$\lfloor w \rfloor = (\alpha(w_1), e_1) \cdot (e_1, \alpha(w_2), e_2) \cdots (e_{n-1}, \alpha(w_n), e_n) \cdot (e_n, \alpha(w_{n+1})).$$

From $\mathcal{F}$-separation to $\mathcal{F}^+$-separation

Main result (separation, generic direction)

If $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathcal{F}$-separable, then L_1 and L_2 are $\mathcal{F}^+$-separable.

Proof (contd.)

$$w = w_1 \cdot w_2 \cdots w_{n+1}$$

where each w_i ends at distinguished position x_i.

$$\lfloor w \rfloor = (\alpha(w_1), e_1) \cdot (e_1, \alpha(w_2), e_2) \cdots (e_{n-1}, \alpha(w_n), e_n) \cdot (e_n, \alpha(w_{n+1})).$$

To a distinguished position x_i in w, associate position $\lfloor x \rfloor = i$ in $\lfloor w \rfloor$.

Lemma

The infix of length $2|S|$ ending at position x in w determines

- whether position x is distinguished,
- the label of the corresponding position $\lfloor x \rfloor$ in $\lfloor w \rfloor$.

Consequence: for $\mathbb{a} \in \mathbb{A}$, there is a formula $\gamma_{\mathbb{a}}(x)$ of $\mathcal{F}^+$ testing that x is distinguished and label of $\lfloor x \rfloor$ is $\mathbb{a}$.

From $\mathcal{F}$-separation to $\mathcal{F}^+$-separation

Main result (separation, generic direction)

If $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathcal{F}$-separable, then L_1 and L_2 are $\mathcal{F}^+$-separable.

Proof (end) If $\mathbb{K} \subseteq \mathbb{A}_\alpha^+$ is $\mathcal{F}$-defined by φ, then there exists an $\mathcal{F}^+$ formula φ^+ over A such that for all $w \in A^+$:

$$w \models \varphi^+ \iff \lfloor w \rfloor \models \varphi.$$

By restricting in φ quantifiers to distinguished positions, and replacing $\mathbb{a}(x)$ by $\gamma_{\mathbb{a}}(x)$.

Finally, if φ defines an $\mathcal{F}$-separator for $\mathbb{L}_1$ and $\mathbb{L}_2$, then φ^+ defines an $\mathcal{F}^+$ separator for L_1 and L_2

Main result, other direction

- Showing that L_1, L_2 $\mathcal{F}$-separability entails $\mathbb{L}_1, \mathbb{L}_2$ $\mathcal{F}^+$-separability relies on Ehrenfeucht-Fraïssé games.
- Example for $\mathsf{FO}^2(<)$.

Conclusion

We shouldn't restrict ourselves to membership

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We shouldn't restrict ourselves to membership, nor to separation.

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We shouldn't restrict ourselves to membership, nor to separation.

- Freezing the framework (to membership or separation) yields limitations.
- This work is just a byproduct of the observation that one can be more demanding on the computed information.
- Generalizing the needed information is often mandatory (see the talk of Thomas P.).

Separation everywhere

Separation everywhere

Heard when preparing these slides on the way

“Attention à la séparation des TGV.”