Mutual Exclusion

Companion slides for
The Art of Multiprocessor Programming
by Maurice Herlihy & Nir Shavit

Mutual Exclusion

- Today we will try to formalize our understanding of mutual exclusion
- We will also use the opportunity to show you how to argue about and prove various properties in an asynchronous concurrent setting

Mutual Exclusion

In his 1965 paper E. W. Dijkstra wrote:

"Given in this paper is a solution to a problem which, to the knowledge of the author, has been an open question since at least 1962, irrespective of the solvability. [...] Although the setting of the problem might seem somewhat academic at first, the author trusts that anyone familiar with the logical problems that arise in computer coupling will appreciate the significance of the fact that this problem indeed can be solved."

Mutual Exclusion

- Formal problem definitions
- Solutions for 2 threads
- Solutions for *n* threads
- Fair solutions
- Inherent costs

Warning

- You will never use these protocols
  - Get over it
- You are advised to understand them
  - The same issues show up everywhere
  - Except hidden and more complex

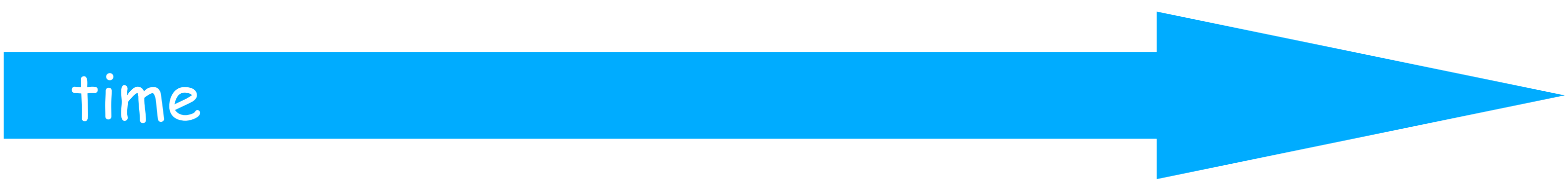
Why is Concurrent Programming so Hard?

- Try preparing a seven-course banquet
  - By yourself
  - With one friend
  - With twenty-seven friends …
- Before we can talk about programs
  - Need a language
  - Describing time and concurrency

Time

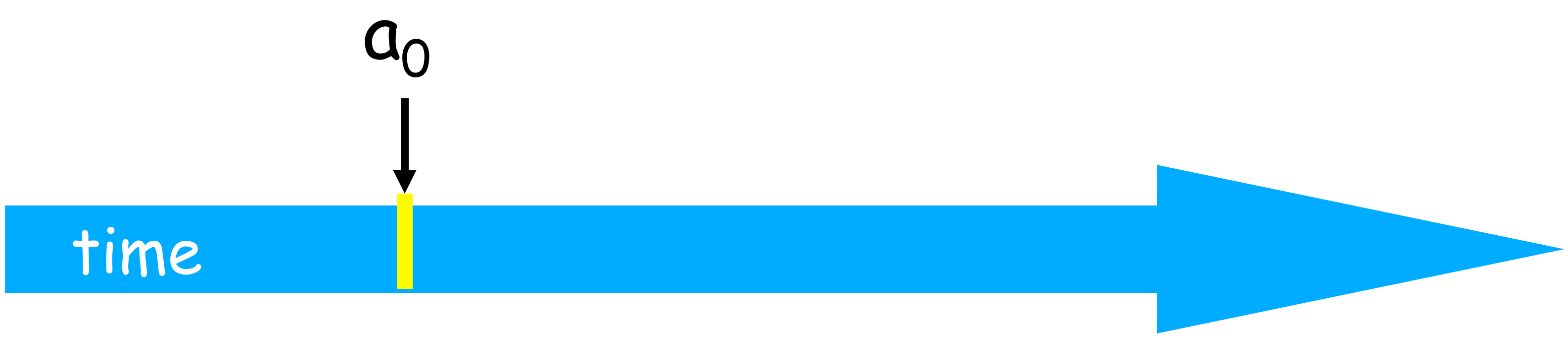
- "Absolute, true and mathematical time, of itself and from its own nature, flows equably without relation to anything external." (I. Newton, 1689)

- "Time is, like, Nature's way of making sure that everything doesn't happen all at once." (Anonymous, circa 1968)



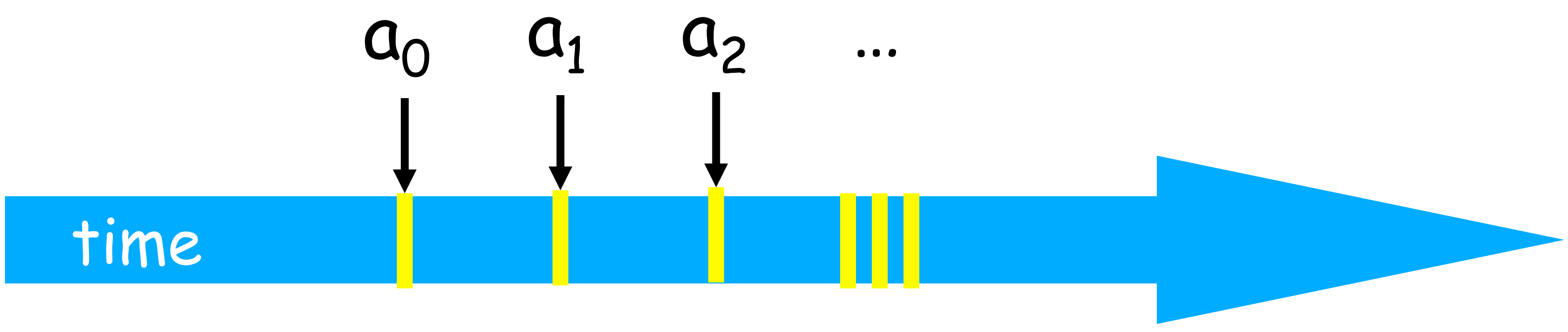
Events

- An *event* a_0 of thread *A* is
  - Instantaneous
  - No simultaneous events (break ties)



Threads

- A *thread* A is (formally) a sequence $a_0, a_1, \ldots$ of events
  - "Trace" model
  - Notation: $a_0 \rightarrow a_1$ indicates order

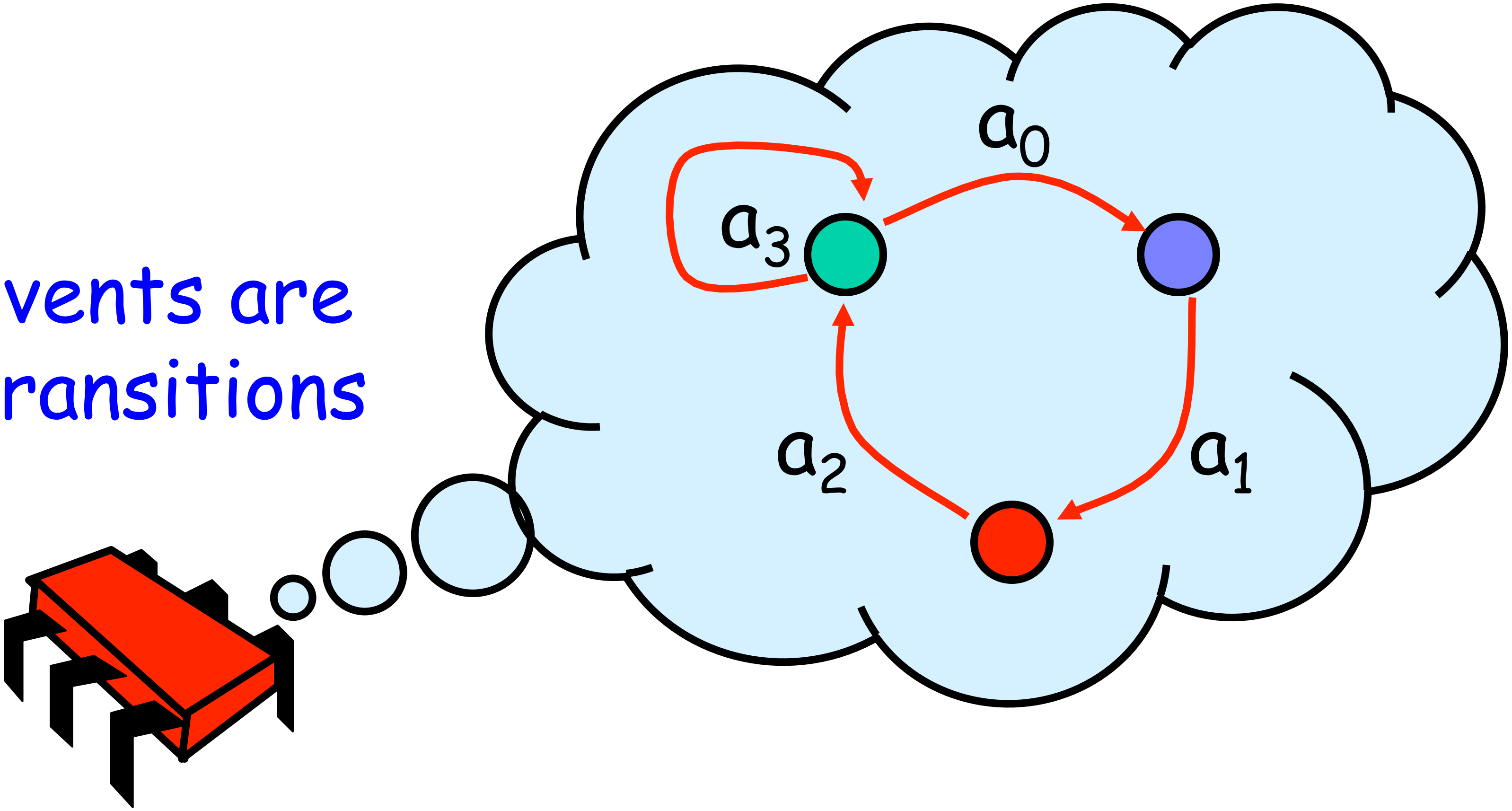


Example Thread Events

- Assign to shared variable
- Assign to local variable
- Invoke method
- Return from method
- Lots of other things …

Threads are State Machines

Events are transitions



States

- Thread State
  - Program counter
  - Local variables
- System state
  - Object fields (shared variables)
  - Union of thread states

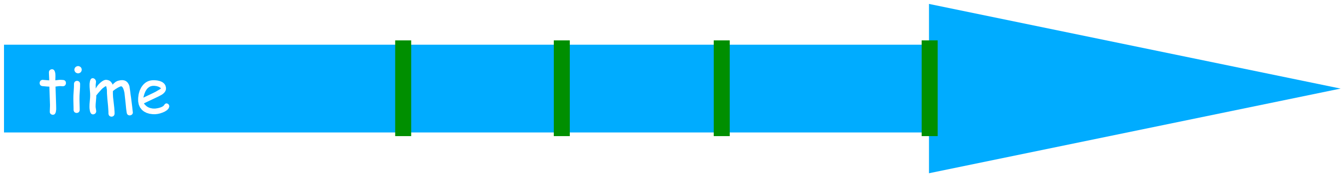
Concurrency

- Thread *A*

Concurrency

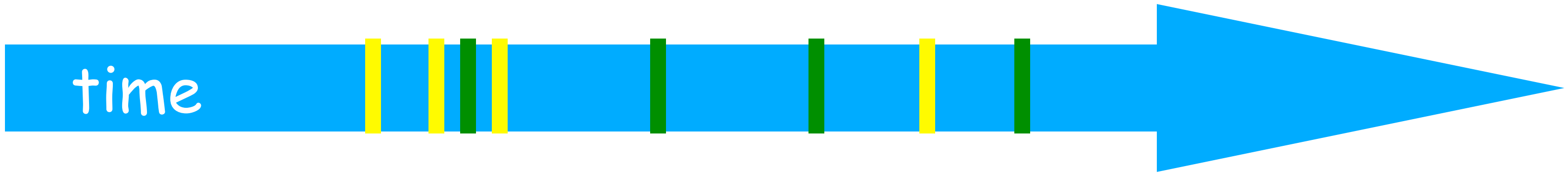
- Thread A

- Thread B



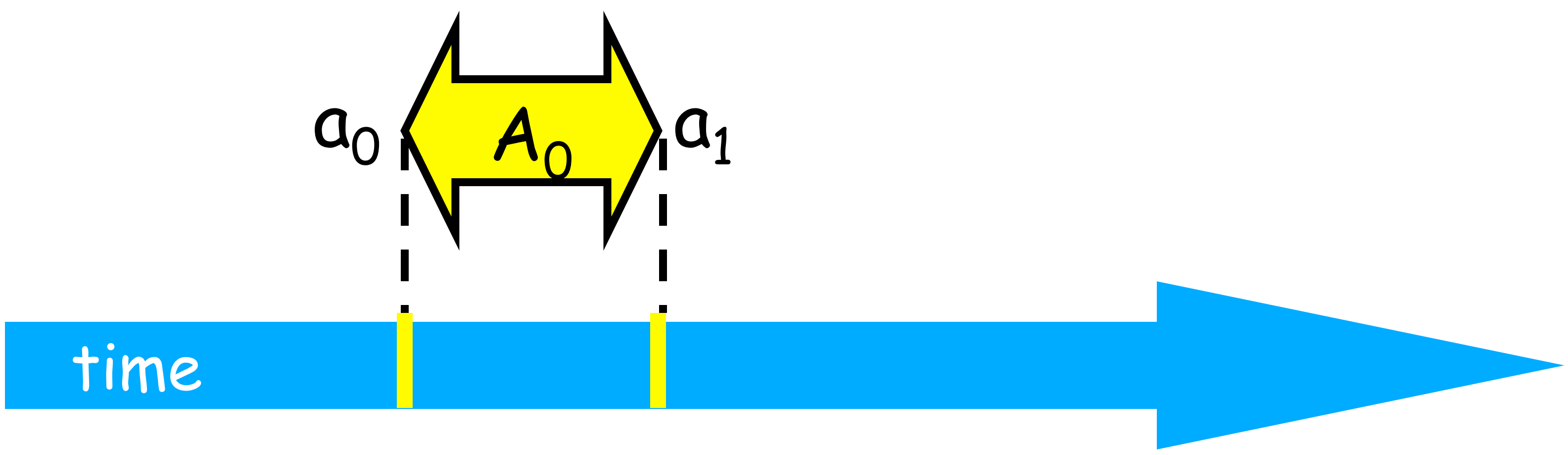
Interleavings

- Events of two or more threads
  - Interleaved
  - Not necessarily independent (why?)

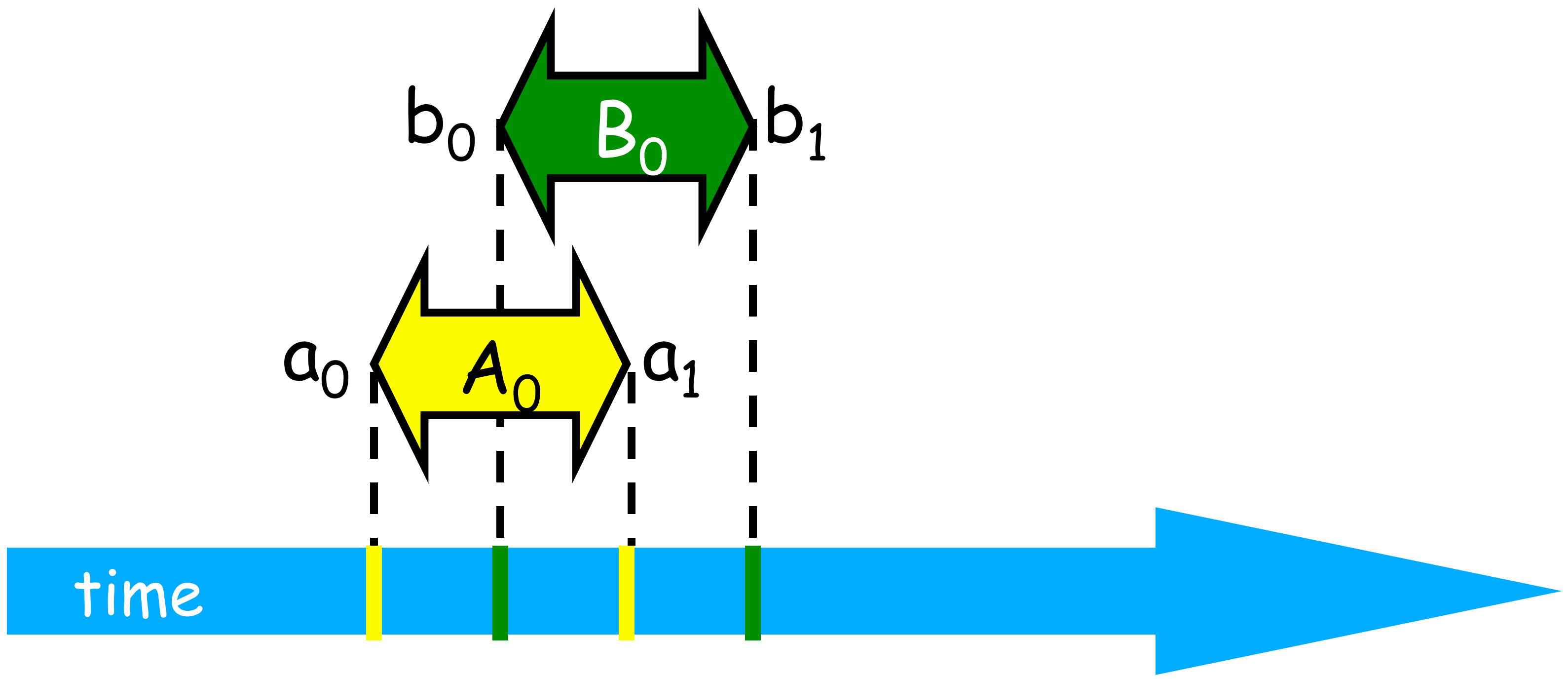


Intervals

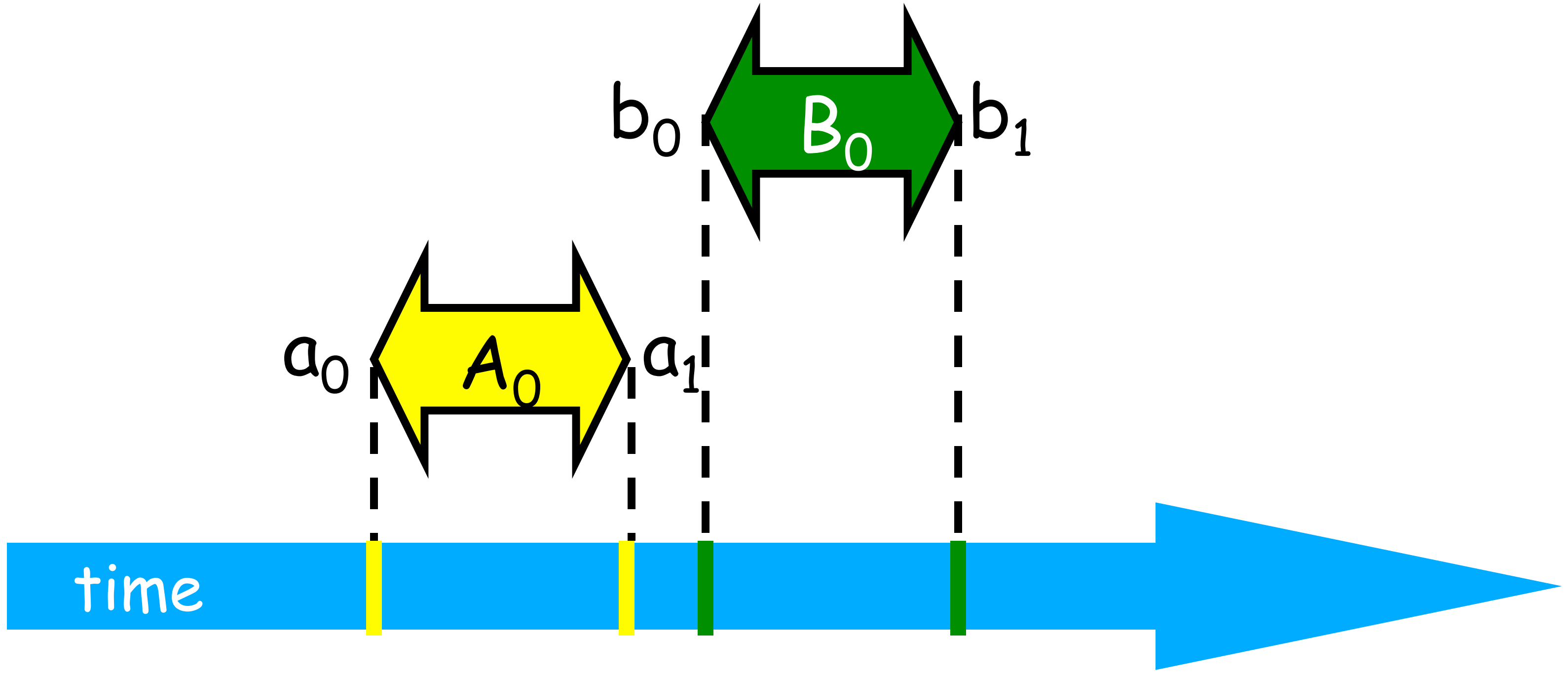
- An *interval* $A_0 = (a_0, a_1)$ is
  - Time between events a_0 and a_1



Intervals may Overlap

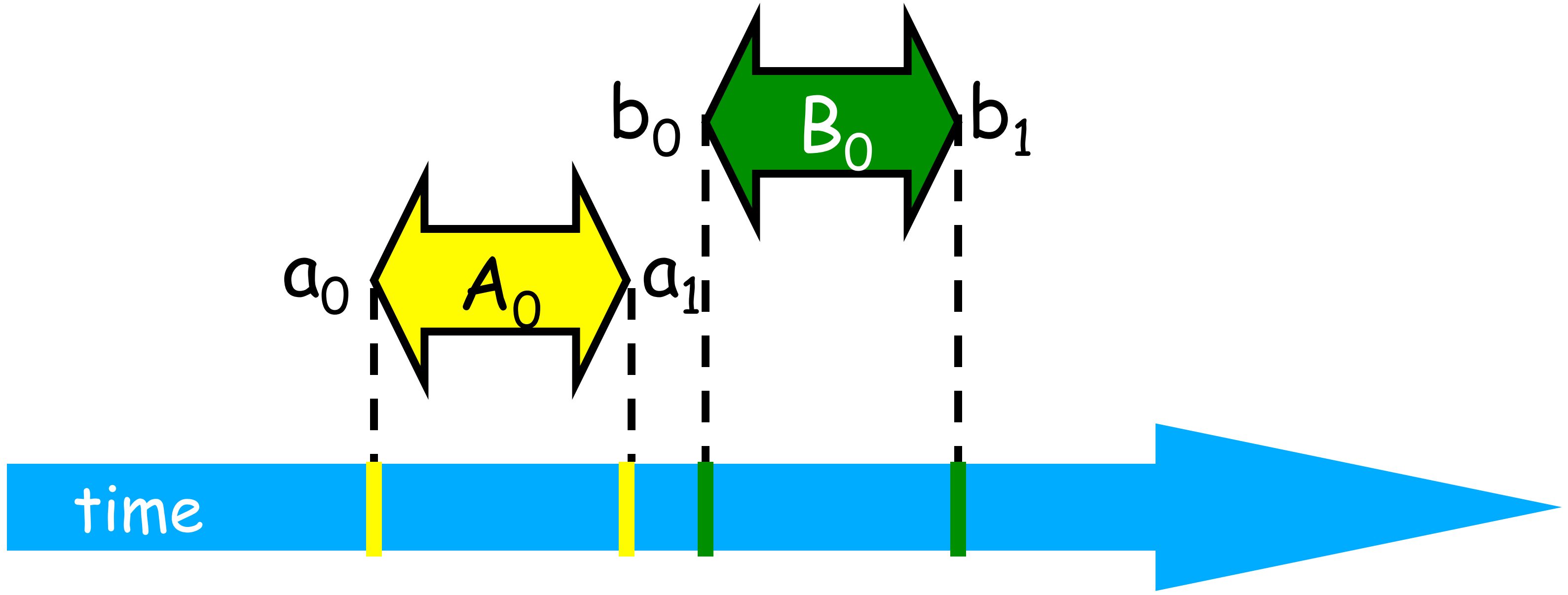


Intervals may be Disjoint

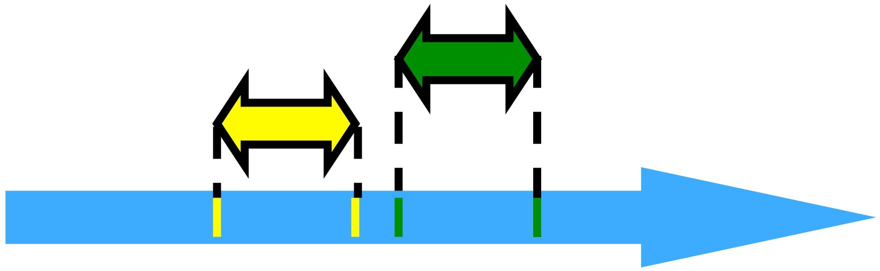


Precedence

Interval A_0 precedes interval B_0

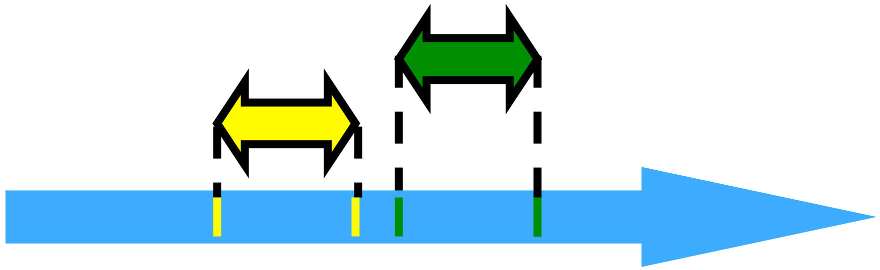


Precedence



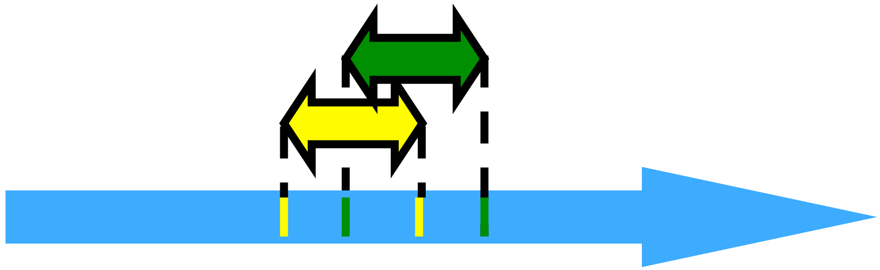
- Notation: $A_0 \rightarrow B_0$
- Formally,
  - End event of A_0 before start event of B_0
  - Also called "happens before" or "precedes"

Precedence Ordering



- Remark: $A_0 \rightarrow B_0$ is just like saying
  - 1066 AD → 1492 AD,
  - Middle Ages → Renaissance,
- Oh wait,
  - what about this week vs this month?

Precedence Ordering



- Never true that A → A
- If A → B then not true that B → A
- If A → B & B → C then A → C
- Funny thing: A → B & B → A might both be false!

Partial Orders
(you may know this already)

- Irreflexive:
  - Never true that $A \rightarrow A$
- Antisymmetric:
  - If $A \rightarrow B$ then not true that $B \rightarrow A$
- Transitive:
  - If $A \rightarrow B$ & $B \rightarrow C$ then $A \rightarrow C$

Total Orders
(you may know this already)

- Also
  - Irreflexive
  - Antisymmetric
  - Transitive
- Except that for every distinct A, B,
  - Either A → B or B → A

Repeated Events

```
while (mumble) {
   a0; a1;
}
```

a_0^k — *k*-th occurrence of event a_0

A_0^k — *k*-th occurrence of interval $A_0 = (a_0, a_1)$

Implementing a Counter

```
public class Counter {
  private long value;

  public long getAndIncrement() {
    temp  = value;
    value = temp + 1;
    return temp;
  }
}
```

Make these steps *indivisible* using locks

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

acquire lock

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

acquire lock

release lock

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

acquire Lock

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Release lock
(no matter what)

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Critical section

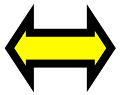
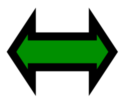
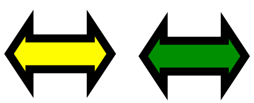
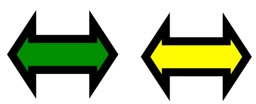
Mutual Exclusion

- Let CS_i^k ⟺ be thread i's k-th critical section execution

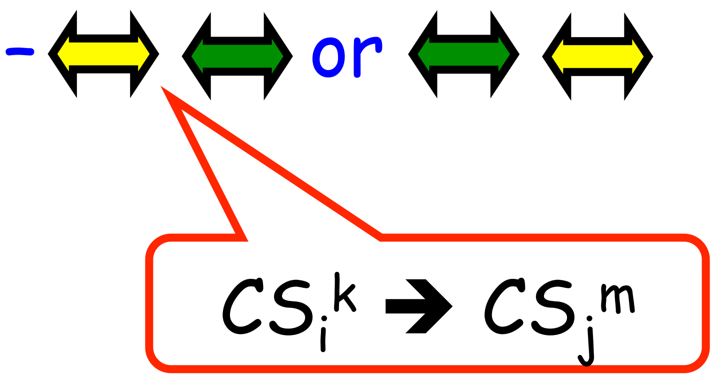
Mutual Exclusion

- Let $CS_i^k \Leftrightarrow$ be thread i's k-th critical section execution
- And $CS_j^m \Leftrightarrow$ be thread j's m-th critical section execution

Mutual Exclusion

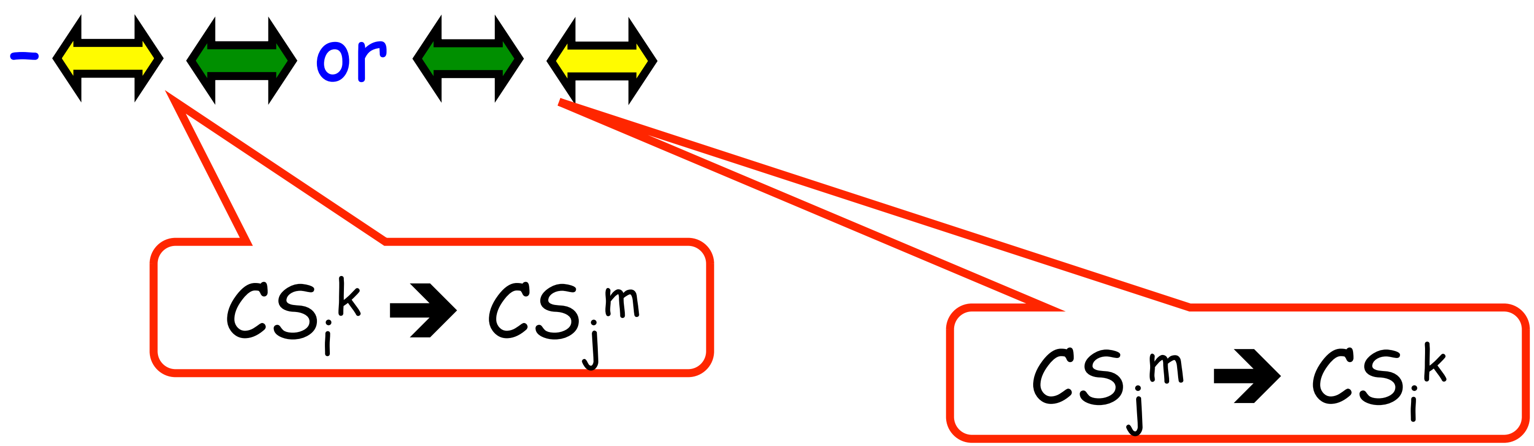
- Let CS_i^k  be thread i's k-th critical section execution
- And CS_j^m  be j's m-th execution
- Then either
  -  or 

Mutual Exclusion

- Let CS_i^k ⇔ be thread i's k-th critical section execution
- And CS_j^m ⇔ be j's m-th execution
- Then either
  - 

Mutual Exclusion

- Let CS_i^k be thread i's k-th critical section execution
- And CS_j^m be j's m-th execution
- Then either



Deadlock-Free

- If some thread calls **lock()**
  - And never returns
  - Then other threads must complete **lock()** and **unlock()** calls infinitely often
- System as a whole makes progress
  - Even if individuals starve

Starvation-Free

- If some thread calls lock()
  - It will eventually return
- Individual threads make progress

Two-Thread vs *n* -Thread Solutions

- Two-thread solutions first
  - Illustrate most basic ideas
  - Fits on one slide
- Then n-Thread solutions

Two-Thread Conventions

```
class … implements Lock {
  …
  // thread-local index, 0 or 1
  public void lock() {
    int i = ThreadID.get();
    int j = 1 - i;
  …
  }
}
```

Two-Thread Conventions

```
class … implements Lock {
  …
  // thread-local index, 0 or 1
  public void lock() {
    int i = ThreadID.get();
    int j = 1 - i;
  …
  }
}
```

Henceforth: i is current thread, j is other thread

LockOne

```
class LockOne implements Lock {
private boolean[] flag =
                               new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

LockOne

```
class LockOne implements Lock {
private boolean[] flag =
                        new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

Set my flag

LockOne

```
class LockOne implements Lock {
private boolean[] flag =
                               new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
  }
```

Set my flag

Wait for other flag to go false

LockOne Satisfies Mutual Exclusion

- Assume CS_A^j overlaps CS_B^k
- Consider each thread's last (j-th and k-th) read and write in the lock() method before entering
- Derive a contradiction

From the Code

- **$write_A(flag[A]=true) \rightarrow read_A(flag[B]==false) \rightarrow CS_A$**

- **$write_B(flag[B]=true) \rightarrow read_B(flag[A]==false) \rightarrow CS_B$**

```
class LockOne implements Lock {
…
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

From the Assumption

- **$read_A$(flag[B]==false) → $write_B$(flag[B]=true)**

- **$read_B$(flag[A]==false) → $write_A$(flag[B]=true)**

Combining

- Assumptions:
  - $\text{read}_A(\text{flag[B]==false}) \rightarrow \text{write}_B(\text{flag[B]=true})$
  - $\text{read}_B(\text{flag[A]==false}) \rightarrow \text{write}_A(\text{flag[A]=true})$
- From the code
  - $\text{write}_A(\text{flag[A]=true}) \rightarrow \text{read}_A(\text{flag[B]==false})$
  - $\text{write}_B(\text{flag[B]=true}) \rightarrow \text{read}_B(\text{flag[A]==false})$

Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
  - $read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
  - $write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$

Combining

- Assumptions:
  - read$_A$(flag[B]==false) $\rightarrow$ write$_B$(flag[B]=true)
  - read$_B$(flag[A]==false) $\rightarrow$ write$_A$(flag[A]=true)
- From the code
  - write$_A$(flag[A]=true) $\rightarrow$ read$_A$(flag[B]==false)
  - write$_B$(flag[B]=true) $\rightarrow$ read$_B$(flag[A]==false)

Combining

- Assumptions:
  - read$_A$(flag[B]==false) $\rightarrow$ write$_B$(flag[B]=true)
  - read$_B$(flag[A]==false) $\rightarrow$ write$_A$(flag[A]=true)
- From the code
  - write$_A$(flag[A]=true) $\rightarrow$ read$_A$(flag[B]==false)
  - write$_B$(flag[B]=true) $\rightarrow$ read$_B$(flag[A]==false)

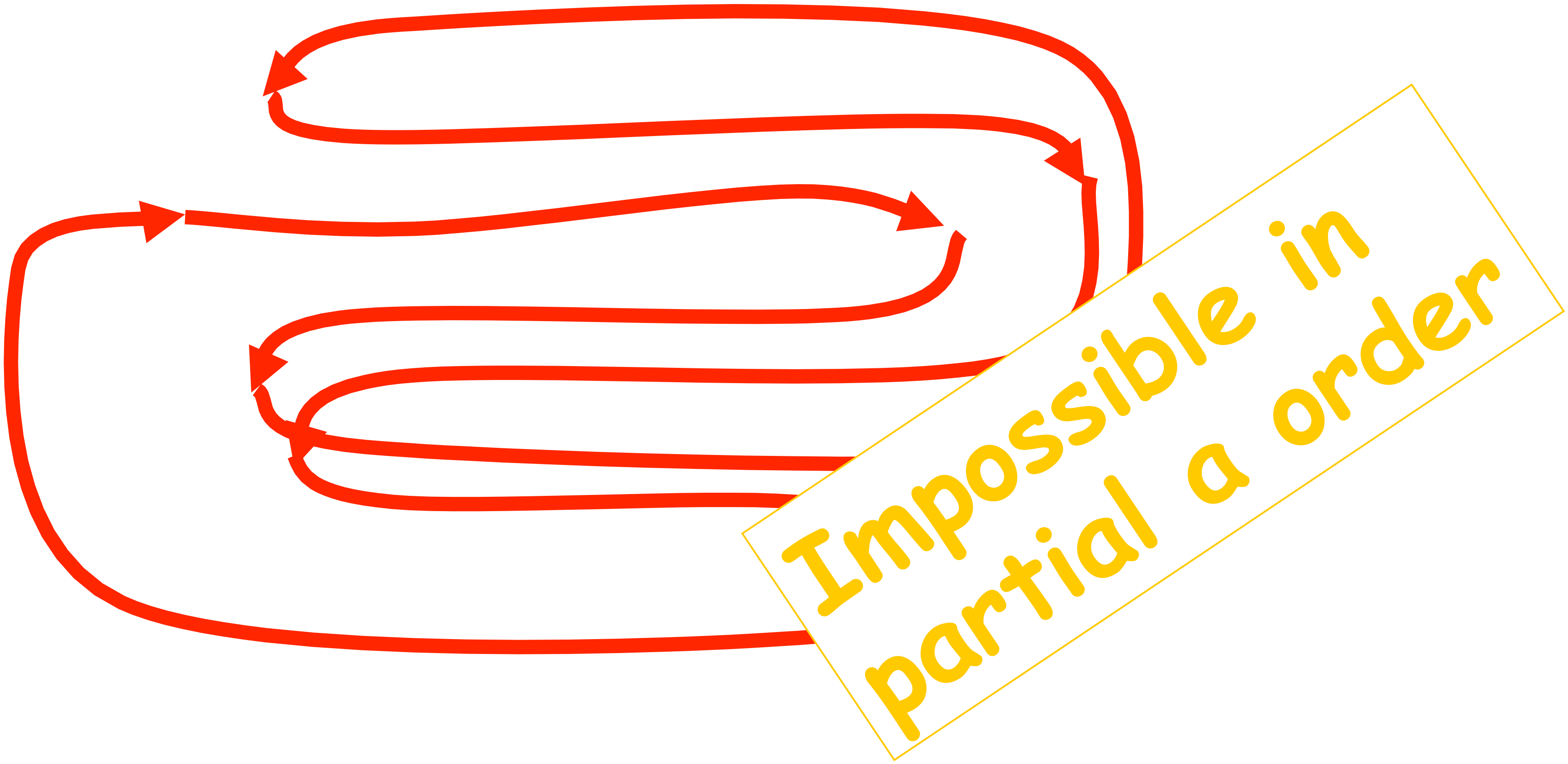
Combining

- Assumptions:
  - read_A(flag[B]==false) → write_B(flag[B]=true)
  - read_B(flag[A]==false) → write_A(flag[A]=true)
- From the code
  - write_A(flag[A]=true) → read_A(flag[B]==false)
  - write_B(flag[B]=true) → read_B(flag[A]==false)

Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
  - $read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
  - $write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$

Cycle!



Deadlock Freedom

- LockOne Fails deadlock-freedom
  - Concurrent execution can deadlock

```
flag[i] = true;        flag[j] = true;
while (flag[j]){}      while (flag[i]){}
```

  - Sequential executions OK

LockTwo

```
public class LockTwo implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

LockTwo

```
public class LockTwo implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Let other go first

LockTwo

```
public class LockTwo implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Wait for permission

LockTwo

```
public class Lock2 implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Nothing to do

LockTwo Claims

- Satisfies mutual exclusion
  - If thread `i` in CS
  - Then `victim == j`
  - Cannot be both 0 and 1
- Not deadlock free
  - Sequential execution deadlocks
  - Concurrent execution does not

```
public void LockTwo() {
  victim = i;
  while (victim == i) {};
 }
```

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Defer to other

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Defer to other

Wait while other interested & I'm the victim

Peterson's Algorithm

```
public void lock() {
  flag[i] = true;
  victim  = i;
  while (flag[j] && victim == i) {};
 }
public void unlock() {
  flag[i] = false;
}
```

Announce I'm interested

Defer to other

Wait while other interested & I'm the victim

No longer interested

Mutual Exclusion

```
public void lock() {
  flag[i] = true;
  victim  = i;
  while (flag[j] && victim == i) {};
```

- If thread 0 in critical section,
  - flag[0]=true,
  - victim = 1

- If thread 1 in critical section,
  - flag[1]=true,
  - victim = 0

Cannot both be true

Deadlock Free

```
public void lock() {
  …
  while (flag[j] && victim == i) {};
```

- Thread blocked
  - only at `while` loop
  - only if it is the victim
- One or the other must not be the victim

Starvation Free

- Thread `i` blocked only if `j` repeatedly re-enters so that

  `flag[j] == true` and `victim == i`
- When `j` re-enters
  - it sets `victim` to `j`.
  - So `i` gets in

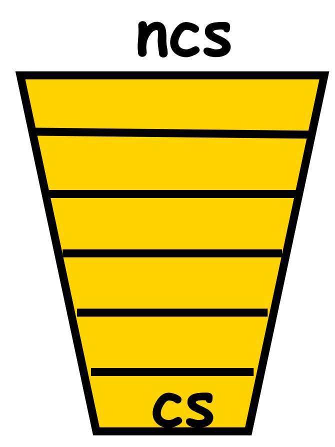
```
public void lock() {
  flag[i] = true;
  victim    = i;
  while (flag[j] && victim == i) {};
}

public void unlock() {
  flag[i] = false;
}
```

The Filter Algorithm for *n* Threads

There are ***n-1*** "waiting rooms" called levels

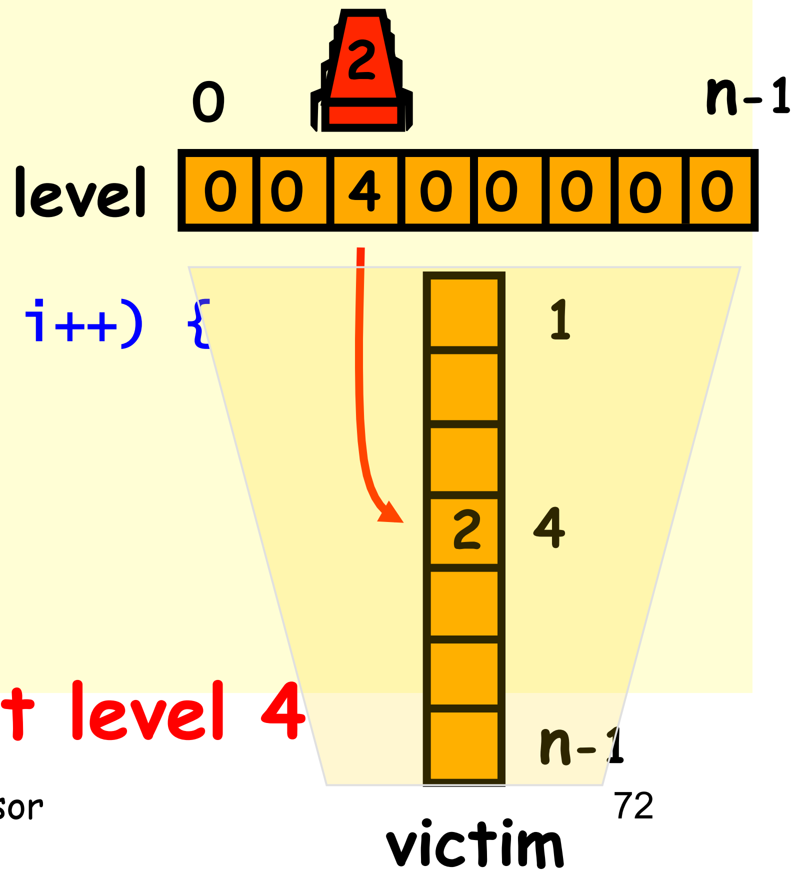
- At each level
  - At least one enters level
  - At least one blocked if many try
- Only one thread makes it through



Filter

```
class Filter implements Lock {
   int[] level;  // level[i] for thread i
   int[] victim; // victim[L] for level L

  public Filter(int n) {
      level  = new int[n];
      victim = new int[n];
      for (int i = 1; i < n; i++) {
          level[i] = 0;
      }}
   …
}
```



Thread 2 at level 4

Filter

```
class Filter implements Lock {
  …

  public void lock(){
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i level[k] >= L) &&
                victim[L] == i );
    }}
  public void unlock() {
    level[i] = 0;
  }}
```

Filter

```
class Filter implements Lock {
  …

  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
                victim[L] == i);
    }}
  public void release(int i) {
    level[i] = 0;
  }}
```

One level at a time

Filter

```
class Filter implements Lock {
  …

  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;

      while ((∃ k != i) level[k] >= L) &&
                 victim[L] == i)
    }}
  public void release(int i)
    level[i] = 0;
  }}
```

Announce intention to enter level L

Filter

```
class Filter implements Lock {
  int level[n];
  int victim[n];
  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i);
    }}
  public void release(int i) {
    level[i] = 0;
  }}
```

Give priority to anyone but me

Filter

Wait as long as someone else is at same or higher level, and I'm designated victim

```
public void lock() {
  for (int L = 1; L < n; L++) {
    level[i]  = L;
    victim[L] = i;
    while ((∃ k != i) level[k] >= L) &&
           victim[L] == i);
  }}
public void release(int i) {
  level[i] = 0;
}}
```

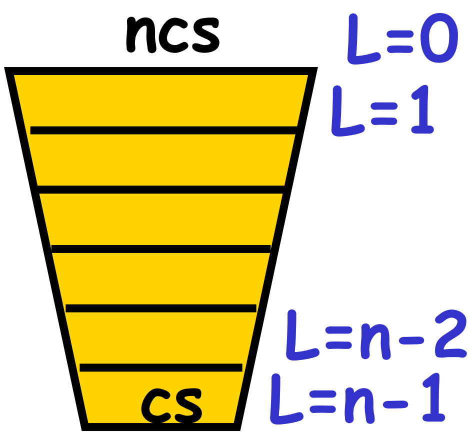
Filter

```
class Filter implements Lock {
  int level[n];
  int victim[n];
  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i);
    }}
```

Thread enters level L when it completes the loop

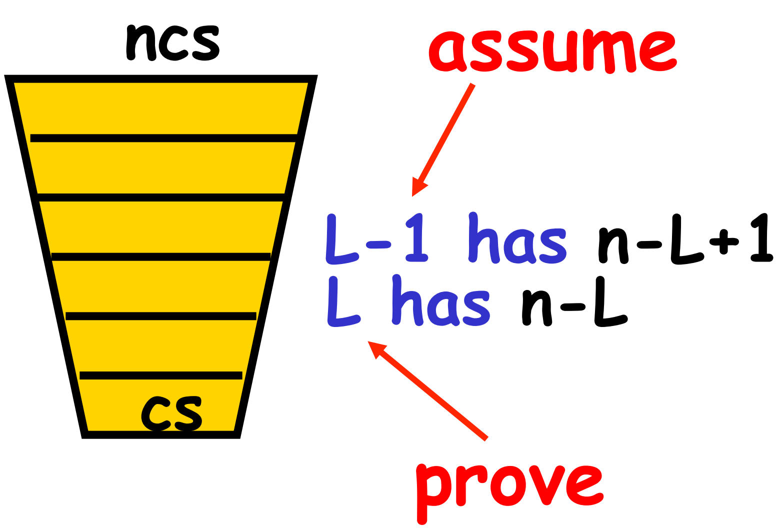
Claim

- Start at level L=0
- At most n-L threads enter level L
- Mutual exclusion at level L=n-1

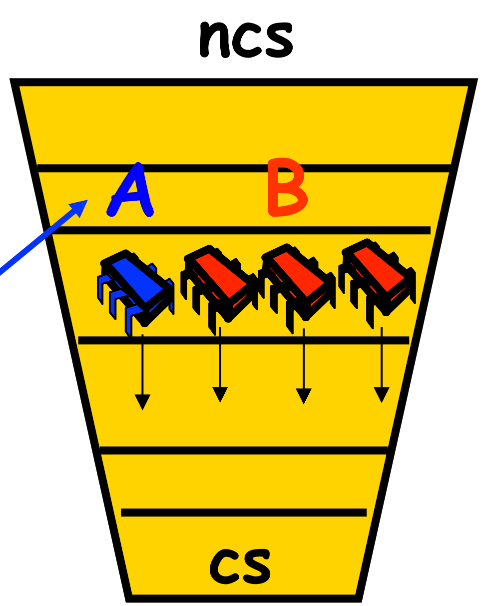


Induction Hypothesis

- No more than n-L+1 at level L-1
- Induction step: by contradiction
- Assume all at level L-1 enter level L
- A last to write victim[L]
- B is any other thread at level L



Proof Structure



Assumed to enter L-1

n-L+1 = 4

n-L+1 = 4

Last to write victim[L]

By way of contradiction all enter L

Show that A must have seen B in level[L] and since victim[L] == A could not have entered

From the Code

(1) $write_B(level[B]=L) \rightarrow write_B(victim[L]=B)$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i] = L;
   victim[L]  = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

From the Code

(2) $\text{write}_A(\text{victim}[L]=A) \rightarrow \text{read}_A(\text{level}[B])$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i] = L;
   victim[L]  = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

By Assumption

(3) $write_B(victim[L]=B) \rightarrow write_A(victim[L]=A)$

By assumption, *A* is the last thread to write **victim[L]**

Combining Observations

(1) $write_B(level[B]=L) \rightarrow write_B(victim[L]=B)$

(3) $write_B(victim[L]=B) \rightarrow write_A(victim[L]=A)$

(2) $write_A(victim[L]=A) \rightarrow read_A(level[B])$

Combining Observations

(1) $write_B(level[B]=L) \rightarrow$

(3) $write_B(victim[L]=B) \rightarrow write_A(victim[L]=A)$

(2) $\rightarrow read_A(level[B])$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i]  = L;
   victim[L] = i;
   while ((∃ k != i) level[k] >= L)
         && victim[L] == i) {};
   }}
```

Combining Observations

(1) $\text{write}_B(\text{level}[B]=L) \rightarrow$

(3) $\text{write}_B(\text{victim}[L]=B) \rightarrow \text{write}_A(\text{victim}[L]=A)$

(2) $\rightarrow \text{read}_A(\text{level}[B])$

Thus, A read level[B] ≥ L,
A was last to write victim[L],
so it could not have entered level L!

No Starvation

- Filter Lock satisfies properties:
  - Just like Peterson Alg at any level
  - So no one starves
- But what about fairness?
  - Threads can be overtaken by others

Bounded Waiting

- Want stronger fairness guarantees
- Thread not “overtaken” too much
- Need to adjust definitions ….

Bounded Waiting

- Divide `lock()` method into 2 parts:
  - Doorway interval:
    - Written D_A
    - always finishes in finite steps
  - Waiting interval:
    - Written W_A
    - may take unbounded steps

r-Bounded Waiting

- For threads A and B:
  - If $D_A^k \rightarrow D_B^j$
    - A's k-th doorway precedes B's j-th doorway
  - Then $CS_A^k \rightarrow CS_B^{j+r}$
    - A's k-th critical section precedes B's (j+r)-th critical section
    - B cannot overtake A by more than r times
- First-come-first-served means r = 0.

Fairness Again

- Filter Lock satisfies properties:
  - No one starves
  - But very weak fairness
    - Not **r-**bounded for any r!
  - That's pretty lame…