Lecture: Consumer Choice

Budget Constraint

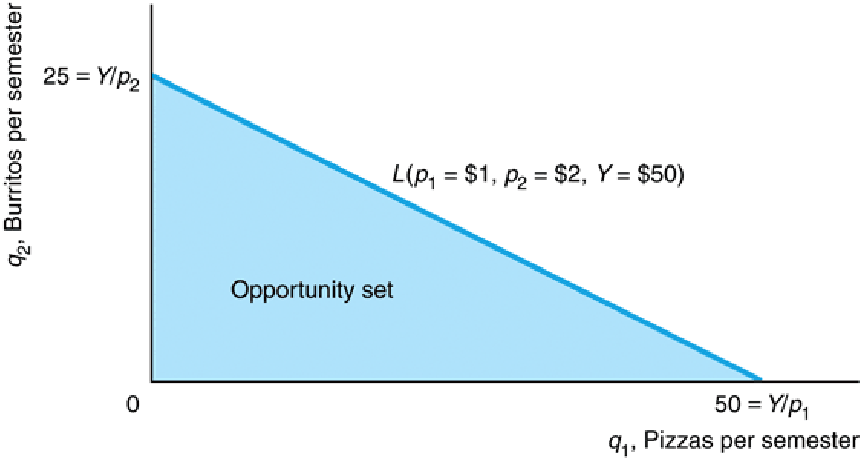
- Suppose that we cannot borrow or save. Then we cannot spend more than our income.
- Assume there are 2 goods, q_1 and q_2, with prices p_1 and p_2, respectively. The available income is Y.
- The set of bundles we can buy is defined by

$$p_1 q_1 + p_2 q_2 \leqslant Y$$
$$q_1 \geqslant 0, \quad q_2 \geqslant 0.$$

- The constraint $p_1 q_1 + p_2 q_2 \leqslant Y$ is known as the **budget constraint**.
- The line with equation $p_1 q_1 + p_2 q_2 = Y$ is known as the **budget line**.
- The set of affordable bundles is called the **feasible set** (also known as the opportunity set).

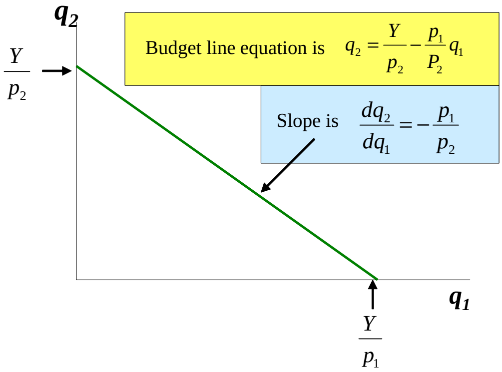
Budget Constraint (Continued)

- Imagine that $p_1 = \$1, p_2 = \$2, Y = \$50$.
- The graph below illustrates the opportunity set.



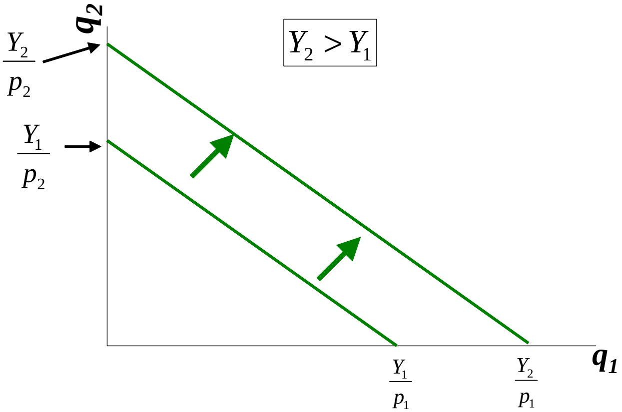
Budget Constraint (Continued)

- We can rewrite the budget line as $q_2 = (Y - p_1 q_1)/p_2$.
- Thus, the slope of the budget line is $-p_1/p_2$.
- The vertical intercept (at $q_1 = 0$) is Y/p_2.
- The horizontal intercept (at $q_2 = 0$) is Y/p_1.



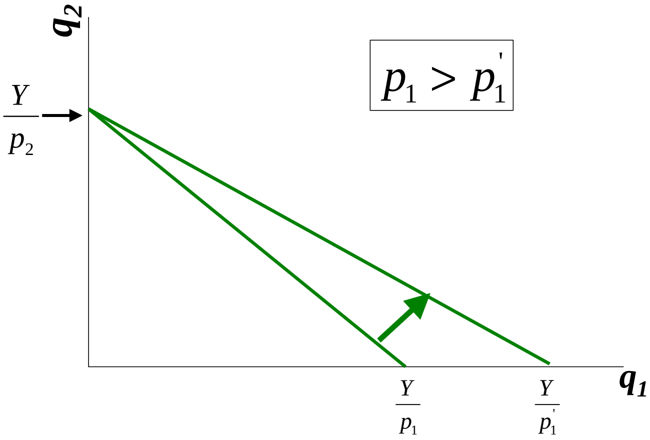
Budget Constraint (Continued)

- If the consumer's budget Y increases while prices stay the same, there will be an outward parallel shift of the budget line.
- This will happen also if all prices fall by the same proportion, while Y remains the same.



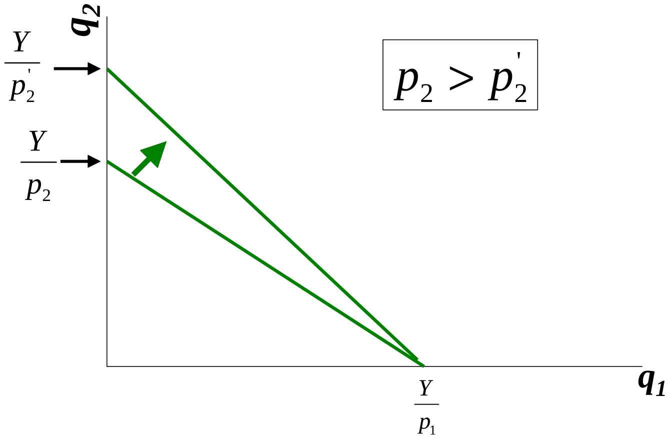
Budget Constraint (Continued)

- If the price of good 1 falls while the price of good 2 and Y remain the same, the budget line rotates outwards around the vertical intercept.



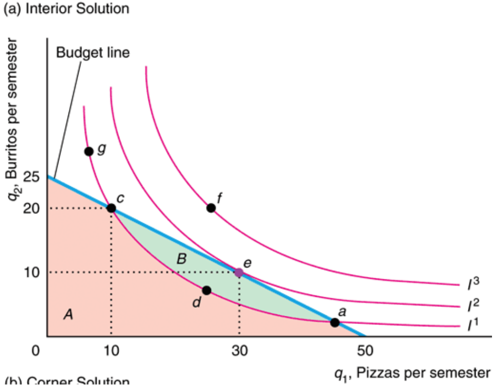
Budget Constraint (Continued)

- If the price of good 2 falls while the price of good 1 and Y remain the same, the budget line rotates outwards around the horizontal intercept.



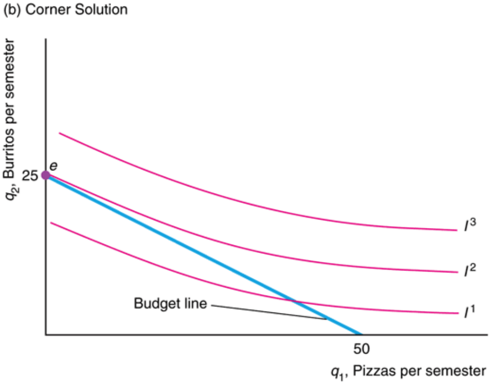
Obtaining the Optimal Bundle Graphically

- Suppose that the consumer wants to choose the bundle which is:
  - feasible (in the opportunity set); and
  - brings the highest utility (lies on the farthest indifference curve).
- If this bundle is **interior** (both quantities are positive), it is given by the point of tangency of the budget line and the indifference curve:



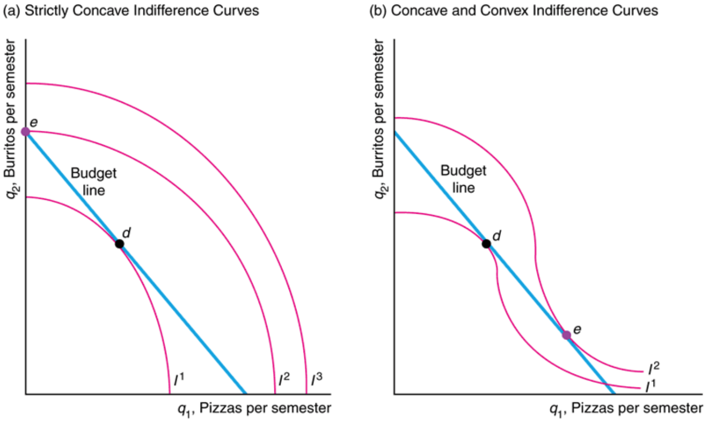
Obtaining the Optimal Bundle Graphically (Continued)

- However, if we have a corner solution (the optimal amount of some good is 0), the indifference curve may not be tangent to the budget line.
- In the graph below, the optimal bundle involves $q_1 = 0$.



Obtaining the Optimal Bundle Graphically (Continued)

- Furthermore, the point of tangency may not give the optimal bundle if preferences are not strictly convex.
- In that case, the optimal bundle may be a corner solution.



Analytical Conditions for Utility Maximization

- Tangency between the budget line and the indifference curve implies that they must have identical slopes: $-\frac{p_1}{p_2} = -\frac{MU_1}{MU_2}$.
- Multiply both sides by -1 to get

$$\frac{p_1}{p_2} = \frac{MU_1}{MU_2}.$$

- Again, this will hold only if the solution is interior!!!
- This condition gives us one equation with two unknowns: q_1 and q_2.
- The other equation needed to pin down the optimal bundle is given by the budget line:

$$p_1 q_1 + p_2 q_2 = Y.$$

- We obtain the utility maximizing bundle by solving these two equations simultaneously.

Interpretation of the Necessary Condition

- Remember the tangency condition

$$\frac{p_1}{p_2} = \frac{MU_1}{MU_2}.$$

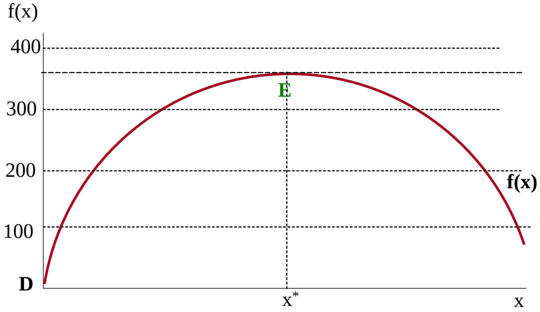
- It can be rewritten as

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}.$$

- MU_1/p_1 is the increase in utility we get if we spend one more dollar on good 1; MU_2/p_2 is the increase in utility we get if we spend on good 2.
- In equilibrium these ratios must be the same!
  - Suppose not, i.e. $MU_2/p_2 > MU_1/p_1$.
  - Then we can attain higher utility without exceeding the budget by taking away \$1 from the consumption of good 1 and spending it on good 2.

Math Aside: Unconstrained Maximization

- Suppose that we have a concave function of a single variable: $y = f(x)$.
- To find the value of x that maximizes f, we solve $df/dx = 0$.



- If f depends on several variables, e.g. $y = f(x_1, x_2)$, to maximize y we solve $\partial f/\partial x_1 = 0$, $\partial f/\partial x_2 = 0$.

Solving the Utility Maximization Problem

- The utility maximization problem is as follows

$$\max_{q_1,q_2} U(q_1,q_2)$$
$$\text{s.t. } p_1q_1 + p_2q_2 = Y.$$

- This problem is constrained: consumers face a budget constraint.
- One way to solve it is to turn it into an unconstrained problem.
- From the budget constraint, we have $q_1 = (Y - p_2q_2)/p_1$. Substitute q_1 in the utility function. The problem becomes

$$\max_{q_2} U\left(\frac{Y - p_2q_2}{p_1}, q_2\right).$$

- We differentiate using the chain rule to obtain the first-order condition

$$-MU_1\frac{p_2}{p_1} + MU_2 = 0.$$

- This can be rewritten as $MU_1/MU_2 = p_1/p_2$, same as before.

The Lagrangian Method

- We can convert our constrained problem into an unconstrained one by constructing an "artificial" **Lagrangian** function.
- That function is constructed as follows: it is equal to our objective function (the utility) plus an extra component: a variable called a Lagrangian multiplier times the "slack" of the budget constraint.

$$\mathcal{L}(q_1, q_2, \lambda) = U(q_1, q_2) + \lambda(Y - p_1 q_1 - p_2 q_2).$$

  - $\mathcal{L}(q_1, q_2, \lambda)$ is the Lagrangian function.
  - $Y - p_1 q_1 - p_2 q_2$ is the slack of the budget constraint (the money left over). At the optimum this slack will be 0.
  - λ is the Lagrangian multiplier. It tells us by how much the maximized utility will increase if the income increases by $1.

- Maximizing the (unconstrained) Lagrangian function with respect to q_1, q_2, λ is equivalent to solving the constrained problem.

The Lagrangian Method (Continued)

- The Lagrangian is $\mathcal{L}(q_1, q_2, \lambda) = U(q_1, q_2) + \lambda(Y - p_1 q_1 - p_2 q_2)$.
- The first-order conditions for q_1, q_2, λ are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_1} &= \frac{\partial U(q_1, q_2)}{\partial q_1} - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial q_1} &= \frac{\partial U(q_1, q_2)}{\partial q_2} - \lambda p_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= Y - p_1 q_1 - p_2 q_2 = 0.
\end{aligned}$$

- The third condition is just the budget constraint: $p_1 q_1 + p_2 q_2 = Y$.
- Rearrange the other first-order conditions:

$$\begin{aligned}
MU_1 &= \lambda p_1 \\
MU_2 &= \lambda p_2.
\end{aligned}$$

- $p_1 > 0, p_2 > 0$; if preferences are monotonic, we can be sure that $\lambda > 0$. Dividing the two conditions yields $MU_1 / MU_2 = p_1 / p_2$.

Example: Cobb-Douglas Utility

- Suppose that $U = q_1^{\alpha} q_2^{1-\alpha}$.
- Let the prices be p_1, p_2 and income be Y.
- Thus, the utility maximization problem is

$$\max_{q_1, q_2} q_1^{\alpha} q_2^{1-\alpha}$$
$$\text{s.t. } p_1 q_1 + p_2 q_2 = Y.$$

- Note that we can rule out corner solutions.
  - If $q_1 = 0$ or $q_2 = 0$, then $U = 0$.
  - If we buy a little of both goods, we get $U > 0$.
- Furthermore, preferences are monotonic: $MU_1 > 0$ and $MU_2 > 0$. So the budget constraint will be binding.
- Therefore, we can use the Lagrangian method.

Example: Cobb-Douglas Utility (Continued)

- The Lagrangian is $\mathcal{L}(q_1, q_2, \lambda) = q_1^{\alpha} q_2^{1-\alpha} + \lambda(Y - p_1 q_1 - p_2 q_2)$.
- The first-order conditions for q_1 and q_2 are

$$\frac{\partial \mathcal{L}}{\partial q_1} = \alpha q_1^{\alpha-1} q_2^{1-\alpha} - \lambda p_1 = 0, \quad \frac{\partial \mathcal{L}}{\partial q_2} = (1-\alpha) q_1^{\alpha} q_2^{-\alpha} - \lambda p_2 = 0.$$

- Rewrite these conditions as

$$\alpha q_1^{\alpha-1} q_2^{1-\alpha} = \lambda p_1, \quad (1-\alpha) q_1^{\alpha} q_2^{-\alpha} = \lambda p_2.$$

- Divide them to get rid of λ:

$$\frac{\alpha q_2}{(1-\alpha) q_1} = \frac{p_1}{p_2} \Longrightarrow p_1 q_1 = \frac{\alpha p_2 q_2}{1-\alpha}.$$

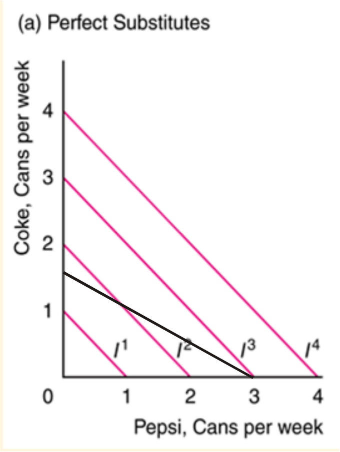
- The other first-order condition is just the budget constraint:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Y - p_1 q_1 - p_2 q_2 = 0.$$

- Substitute $p_1 q_1$: $Y - \frac{p_2 q_2}{1-\alpha} = 0$. Thus, $q_2 = \frac{(1-\alpha)Y}{p_2}$, $q_1 = \frac{\alpha Y}{p_1}$.

Example: Linear Utility

- Suppose that the utility is $U = aq_1 + bq_2$.
- The indifference curves will be straight lines with a slope of $-a/b$.
- We get a corner solution, so we cannot use the Lagrangian method.

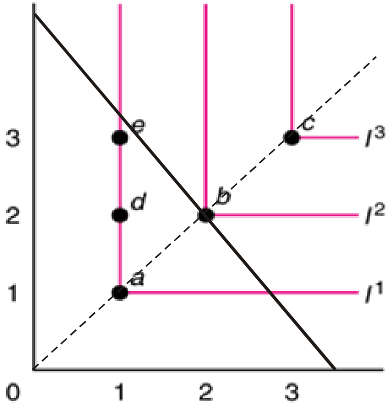


Example: Linear Utility (Continued)

- The best way to solve this problem is to use a diagram.
- The slope of the indifference curves is $-a/b$.
- The slope of the budget line is $-p_1/p_2$.
- If the budget line is flatter than the indifference curves, $q_2 = 0$.
  - The budget line is flatter when $-p_1/p_2 > -a/b \Longrightarrow p_1/p_2 < a/b$.
  - In that case, we spend all our income on q_1: $q_1 = Y/p_1$.
- If the budget line is steeper than the indifference curves, $q_1 = 0$.
  - The budget line is steeper when $p_1/p_2 > a/b$.
  - In that case, we only buy q_2: $q_2 = Y/p_2$.
- If $p_1/p_2 = a/b$, then all bundles on the budget line bring the same utility.

Example: Leontief Utility

- Suppose that the utility is $U = \min\{aq_1, bq_2\}$.
- In this case, indifference curves have a kink.
- The solution is interior but we cannot use the Lagrangian method, because the slope of the indifference curves at the kink is not defined!

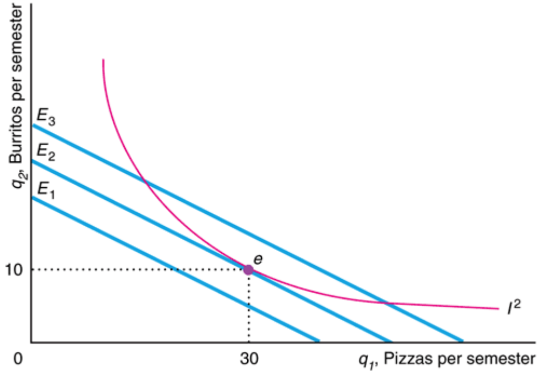


Example: Leontief Utility (Continued)

- Again, the best way to solve this problem is graphically.
- Obviously, the optimal bundle will be at one of the kinks.
- The kinks of the indifference curves lie on a ray from the origin whose equation is
  $$aq_1 = bq_2.$$
  This gives us one equation.
- The other equation is the budget constraint:
  $$p_1 q_1 + p_2 q_2 = Y.$$
- Solving the equations simultaneously, we get
  $$q_1 = \frac{Y}{p_1 + p_2 a/b} \quad q_2 = \frac{Y}{p_2 + p_1 b/a}.$$

Expenditure Minimization

- Now instead of maximizing utility subject to a budget constraint, we will consider a different problem.
- Suppose that the consumer's objective is to minimize his expenditure, subject to attaining a certain level of utility.
- This *dual* problem is known as **expenditure minimization**.



Expenditure Minimization (Continued)

- The expenditure minimization problem is formulated as follows:

$$\min_{q_1,q_2} p_1q_1 + p_2q_2 \quad \text{s.t. } U(q_1,q_2) = \bar{U}.$$

- Again, we can solve it with the Lagrangian method:

$$\mathcal{L}(q_1,q_2,\lambda) = p_1q_1 + p_2q_2 + \lambda(\bar{U} - U(q_1,q_2)).$$

- The first order conditions are

$$\begin{aligned} \partial\mathcal{L}/\partial q_1 &= \lambda p_1 - \partial U(q_1,q_2)/\partial q_1 = 0, \\ \partial\mathcal{L}/\partial q_2 &= \lambda p_2 - \partial U(q_1,q_2)/\partial q_2 = 0, \\ \partial\mathcal{L}/\partial \lambda &= \bar{U} - U(q_1,q_2) = 0. \end{aligned}$$

- From the first two first-order conditions, we get the familiar equation $MU_1/MU_2 = p_1/p_2$. The second equation is just the constraint: $U(q_1,q_2) = \bar{U}$. We solve them to get q_1, q_2.

Expenditure Minimization: Cobb-Douglas Utility

- Suppose that $U = q_1^\alpha q_2^{1-\alpha}$. The Lagrangian is thus given by

$$\mathcal{L}(q_1, q_2, \lambda) = p_1 q_1 + p_2 q_2 + \lambda(\bar{U} - q_1^\alpha q_2^{1-\alpha}).$$

- The condition $MU_1/MU_2 = p_1/p_2$ takes the same form as in the utility maximization example:

$$\frac{\alpha q_2}{(1-\alpha)q_1} = \frac{p_1}{p_2} \Longrightarrow q_1 = \frac{\alpha p_2}{(1-\alpha)p_1} q_2$$

- Now the constraint is $q_1^\alpha q_2^{1-\alpha} = \bar{U}$.
- Substitute q_1 in the constraint to get

$$q_2 = \bar{U}\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^\alpha, \quad q_1 = \bar{U}\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}.$$

Expenditure Function

- We can substitute the solutions q_1, q_2 of our expenditure minimization problem in the consumer's expenditure $p_1 q_1 + p_2 q_2$.
- This will give us the *minimized* expenditure of this consumer as a function of prices p_1, p_2 and the utility target $\bar{U}$.
- This expression is known as the **expenditure function**.
- In our Cobb-Douglas example, the expenditure function is

$$E(p_1, p_2, \bar{U}) = p_1 \bar{U} \left(\frac{\alpha p_2}{(1-\alpha) p_1} \right)^{1-\alpha} + p_2 \bar{U} \left(\frac{(1-\alpha) p_1}{\alpha p_2} \right)^{\alpha}.$$

- After some algebra, we get

$$E(p_1, p_2, \bar{U}) = \bar{U} \left(\frac{p_1}{\alpha} \right)^{\alpha} \left(\frac{p_2}{1-\alpha} \right)^{1-\alpha}.$$