

**AN INTORDUCTION TO HAMILTON-JACOBI
THEORY**

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1. THE INTRODUCTION

One of the major things in Classical Dynamics is to solve for the systems Hamilton's equations which tells out everything about the system. This theory deals with one of the beautiful methods of solving (infact, a general one, because this is the formal method for time dependant Hamiltonian). *Our idea* will be to seek a Canonical transformation of the coordinates (q, p) to a new set of coordinates that are cyclic in nature. The equations of transformation relating the old to the new canonical variables are then exactly the solution to the mechanical problem.

2. A GLANCE AT FEW TERMS THAT APPEARED ABOVE

2.1. The Hamilton's Principle. The motion of a system from time t_1 to t_2 is such that the integral $I = \int_{t_1}^{t_2} L dt$, has a stationary value for the correct path, where $L = p\dot{q} - H$.

2.2. The Hamilton's equations of Motion. Say the Hamiltonian is H and the phase space coordinates are (q, p) , the the Hamiltonian's equations of motion are

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i\end{aligned}$$

These are basically the defining equations of any Dynamical Systems.

2.3. Canonical Transformation. A canonical transformation is a map from old coordinates to new coordinates so that the solution to the problem will be easier in new coordinates. The transformation should satisfy the following property. Say, our transformed Coordinates are (Q, P) and there exists a function K (called the transformed Hamiltonian) such that they satisfy the Hamilton's equations of motion i.e.

$$\begin{aligned}\frac{\partial K}{\partial P_i} &= \dot{Q}_i \\ \frac{\partial K}{\partial Q_i} &= -\dot{P}_i\end{aligned}$$

2.4. Transformation Theory. We can see that the Action integral (the variation of the Hamilton's principle function written above) modifies for the new coordinates as $\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t)) dt = 0$. From the new and old equations we cannot conclude that the integrands are equal because the equations only tells us that it has zero variation at the end points. The solution of this is

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K - \frac{dF}{dt}$$

Here the function F is called the generating function. λ is only the scaling transformation and for suitable choice we can make $\lambda = 1$.

2.5. Cyclic coordinates and their implication for us. They are defined as those coordinates that do not appear in the statement of Hamiltonian. Since these are absent, conjugate momentum are constant. So, if our new coordinates are cyclic, along with our new Hamiltonian, will be easier to work with :

- All momentum will be constant.
- K is constant. Therefore, the time derivatives of coordinates are constant.
- So, the individual motion would each be linear in time.

If K is constant, then we can always choose it to be 0. Hence,

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0$$

$$\frac{\partial K}{\partial Q_i} = -\dot{P}_i = 0$$

3. HAMILTON-JACOBI EQUATION

From above two equations and also from the relation $p_i \dot{q}_i - H = P_i \dot{Q}_i - K - \frac{dF_2(Q,p,t)}{dt}$ and $K = 0$ (expand the total derivative of F_2) we get

$$K = H + \frac{\partial F_2}{\partial t} \Rightarrow H + \frac{\partial F_2}{\partial t} = 0$$

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}$$

Now, if we substitute using the second relation in to the first equation we will get as follows

$$H(q_1, \dots, q_n; \frac{\partial F_2}{\partial q_1}, \dots, \frac{\partial F_2}{\partial q_n}; t) + \frac{\partial F_2}{\partial t} = 0$$

This is the **Hamilton Jacobi Equation**. It is customary to call the generating function as S and call it *Hamilton's Principal Function*.

4. SO WHAT'S NEXT?

Let's assume that there exists solution of form $S = S(q_1, \dots, q_n; \alpha_1, \dots, \alpha_{n+1}; t)$ for the HJ equation. This is in fact not the only solution but one of the possibly many. These kind of solutions are called complete solutions of first order partial differential equation. One of the α 's (these are basically constants) is not important at all for us as it is an additive constant of S . Hence we can drop without loss of generality α_{n+1} . Hence the left over constants are not purely additive (assume!). We

can see that S tallies with F_2 ! Hence, we can choose our α_i to be P_i which are indeed constant as we chose them to be cyclic. Hence,

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i}$$

At $t = t_0$, we get n equations relating the n α 's with the initial q and p values, helping us in evaluating the constants of integration in terms of specific initial conditions of our given problem. We can also get

$$Q_i = \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}$$

From here β_i can be calculated by solving it at $t = t_0$. Hence we can again rearrange to get $q_i = q_i(\alpha, \beta, t)$ and with the help of above expression for p_i and this expression we can get $p_i = p_i(\alpha, \beta, t)$.

Hence, we indeed got the expression for q and p which is the sole aim of our problem in classical mechanics. Hence, as we solve the HJ equation we get the solution for the mechanical problem.

5. PHYSICAL SIGNIFICANCE OF S

Let's consider the differential of S with respect to t

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial P_i} \dot{P}_i + \frac{\partial S}{\partial t} \\ \text{but, } \dot{P}_i &= 0; \dot{p}_i = \frac{\partial S}{\partial q_i}; H + \frac{\partial S}{\partial t} = 0 \\ &\Rightarrow \frac{dS}{dt} = p_i \dot{q}_i - H \\ &\Rightarrow \frac{dS}{dt} = L \\ &\Rightarrow S = \int_{t_1}^{t_2} L dt \end{aligned}$$

Hence, we see that S is in fact like the Action.