GENERALIZED COORDINATES

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Abstract. This primarily deals with the conception of phase space and the uses of it in classical dynamics. Then, we look into the other fundamentals that are required in dynamics before we start with perturbation theory i.e. topics like generalised coordinates and genral analysis of dynamical systems.

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The general analysis of dynamics using the conception of phase space is a very deep physical aspect explained in mathematical structure. Let's intially look what is a phase space and the requirement of a phase space. Well before that let's see the use of generalized coordinates.

1. GENERALIZED COORDINATES

1.1. **Why generalized coordinates?** *I* have not particularily used the vector symbol over x. It is implicity asssumed to be so and when I speak of components, then I use it with an index (either a subscript or supeerscript)

We always try to select coordinates such that they have a visualizable geometric significance. They are mostly chosen on the basis (being independent usually helps us). They are system dependent unlike the usual coordiantes. Such coordinates are helpful principally in Lagrangian Dynamics, where the forms of the principal equations describing the motion of the system are unchanged by a shift to generalized coordinates from any other coordinate system.

1.2. **An approach mentioned.** Say we have a N particle system with K holonomic constraint(i.e. constraints of the form $f_i(x_1, x_2, ...x_N, t) = 0$ and $i = 1, 2, ...K < 3N$) on the system, where, x_j are position vectors of N particles.

Now, let's consider a one-particle system. Our surface is $f(x, t) = 0$ and from Newton's equations we have $m\ddot{x} = \vec{F} + \vec{C}$ where $\vec{C}$ is the unknown force of constraint. So, we have 6 unknwn things but only 4 equations.

Our, SOS call has been answered by the considering that the force of constraint is normal to th surface at every point. Because, the only effect by the parralel forces (if they have any) is that they increase the acceleration along the surface. Now, how can we choose this force. We know that $\nabla f(\vec{x})$ is always perpendicular to the surface unless $\nabla f = 0$. So, let's leave out this case in our analysis. (This can be mathematically thought as,if K¿1, N¿1, the requirement is that matrix of $\frac{\partial f_i}{\partial x^\alpha}$ elements are atleast of rank K). Hence the constraint force can be written as

$$\vec{C} = \lambda(t)\nabla f(\vec{x}, t)$$

Hence, we have 4 equations and 4 unknowns as $\vec{C}$ has reduced to be dependent on λ which is dependent only on time.

*P*hysical reason why $\vec{C}$ should be perpendicular to the surface is that forces of constriant do no wrok other than keep reduce the availability of states to the system i.e. keep the particle confined to the surface. (Here, we do not consider the forces lik friction which are dissipative i.e our system become non-conservative). Such surfaces are called "smooth".

Hence we have,

$$m\ddot{x} = \vec{F} + \lambda(t)\nabla f(\vec{x}, t)$$

Now, let's try to eliminate $\lambda(t)$. As $\lambda\nabla f$ is perpendicular to the surrface, so we can eliminate by taking the component tangential to plane. Let $\vec{\tau}$ be an arbitrary vector tangential to the surface at $\vec{x}$ at, i.e. only restriction on $\tau.\nabla f = 0$ Hence, we have

$$(m\ddot{x} - \vec{F}).\tau = 0$$

This implies $m\ddot{x} - \vec{F}$ is perpendicular to surface at $\vec{x}$ and at time t. We can find such τ at each x. As τ is arbitrary vector tangential to surface, there are two linearly independent vectors at each $\vec{x}$ and hence two linearly independent vector functions of $\vec{x}$ and t. Hence we get two equations for the above equation. So now we have 3 equations and similarily 3 unknowns.

Now, let's generalize this above procedure for N particles and K constraints. (We do NOT use the summation notation below). Equation for the ith particle is

$$m_i\ddot{\vec{x}}_i = \vec{F}_i + \vec{C}i$$

We take $\vec{C} = \sum_{j=1}^{k} \lambda_j \nabla_i f_i$ where ∇_i is the gradient with respect to $\vec{x}_j$ and λ_j and are K in number.

Now, if $\frac{\partial V}{\partial t} = 0$. As we shall see later, $\frac{dE}{dt} = \sum_j \frac{\partial f_j}{\partial t}$.

Let $\vec{\tau}$ be N arbitrary vectors tangential to the surface. and hence wa have that

$$\sum_{i=1}^{N} \vec{\tau_i}.\nabla_i f_j$$

Compoents of N vectors $\tau_i \Rightarrow 3N - K$ components are independent. Then we have

$$(m_i\ddot{x}_i - \vec{F}_i.\vec{\tau_i} = 0$$

The above principle is called that D'Alembert's Principle. By the same arguments as with the one particle case, we get that $3N - K$ componenets of $\vec{\tau_i}$ lead to $3N - K$ relations and the K constraint equations, totally counting to $3N$ equations and our unknows are the $3N$ unknown components of x_i. Hence,can be solved.

By doing all the above things, our problem has finally reduced to finding a suitable $\vec{\tau}$ such that the "only" condition on it is satisfied. In particle system, it is any arbitrary vector tangential to the surface. Here, in N particles, it is not easy to visualise (we will later come to know that this is nothing but a vector on the Tangent manifold $T^*\mathbb{Q}$). But, we know tha tthe constraint equations define a $3N - K$ dimensional hypersurface in the $3N$ dimensional $\mathbb{E}^{3N}$. We call this Configuration Hyperspace or Configuration Manifold.

We know the mathematical condition on $\vec{\tau}$. This defines a vector in $\mathbb{E}^{3N}$ which is a kind of tangent vector to the configuration manifold. Hence, it defines a 3-vector tangential to $f = 0$ in one particle system.

Similarily we get a $3N$-vector tangential to $f_j = 0$ hypersurface ($\mathbb{Q}$). Hence, picking tau_i means picking tangent vector in the $3N$ dimensions.

Let's define q^α in the $3N$ space for which $\mathbb{Q}$ is the configuration hypersurface. Consider a region of $\mathbb{E}^{3N}$ that has x_i of $\mathbb{Q}$ and let q^α where $\alpha = 1, 2, ..., 3N$, be the new coordiantes of the region such they are invertible with the x_i.

$$q^\alpha = q^\alpha(x_1, ...x_N, t)$$

And these ar etwice differentible because they sould represent the acceleration also. Our objective is to pick them such that equations of constraints becomes trivial (i.e some of the q^α are constant). When we write the equation of motion in these new coordinates those coordinates drop out which are constant. This an be don by choosing q^α such that K of them depend on$\vec{x_i}$ through function appearing in the constraint equations. Let the last K be of this kind. Hence, we have

$$q^{n+j}(\vec{x}) = R_j(f_1(\vec{x}), ..., f_K(\vec{x})); j = 1, ...K$$

where we represent $\vec{x}$ for $\vec{x_i}$ and $n = 3N - K$ (dimension of $\mathbb{Q}$. These are the last K equations of 3N equations that give q^α in x_i, and they too must be invertible, which means that it must be possible to solve them for $f_j = f_j(q^{n+j}, ..., q^{n+K})$. When the constraint conditions are imposed, they force the last K of the new coordinates to be constants.

$$q^{n+j} = R(0, ...0)$$

This is what we mean by telling that the constraints are trivial in these coordiantes.

Now, it's all about the dependance of the remaining q^α on time. q^α can be used as well as x_i to define a point in $\mathbb{E}^{3N}$. The last equation requires the point to lie in $\mathbb{Q}$. Hence, q^α form the coordinate system on $\mathbb{Q}$.

1.3. **A small digression which I made in the talk that was confusing.** We have considered $\vec{F} = -\nabla V(x, t)$ and taking hte dot product with our equation we get

$$m.\ddot{x}.\dot{x} \equiv \frac{d}{dt}\{\frac{1}{2}m\dot{x}^2 + V\} = -\nabla V.\vec{x} + \lambda \nabla f.\dot{\vec{x}}$$

Let $\vec{x(t)}$ be a solution of equation of motion. Any particle remains on the surface for $f(\vec{x}(t), t) = 0 \Rightarrow \frac{df}{dt} = 0$. But we know

$$\frac{df}{dt} = \nabla f.\dot{\vec{x}} + \frac{\partial f}{\partial t}$$
$$\frac{dV}{dt} = \nabla V.\dot{\vec{x}} + \frac{\partial V}{\partial t}$$

Hence we have,

$$\frac{dE}{dt} = \frac{d}{dt}\{\frac{1}{2}m\dot{x}^2\} = \frac{\partial V}{\partial t} - lambda\frac{\partial f}{\partial t}$$

Hence, the total energy of the particle changes only if V or f are explicit functions of time (i.e if the potential depends on time or the surface is moving). These are already removed from our cases and analysis.