Pendulum with a vibrating base

Chennai, March 14, 2006

1 Fast perturbations- rapidly oscillating perturbations

We consider perturbations of a Hamiltonian perturbed by rapid oscillations. Later we apply this to the case of a pendulum with a vibrating base. The basic idea is to see if the unstable fixed point of the *mathematical pendulum* can be made stable by perturbing with rapid oscillations of the base (pivot)[1]. We first consider the general theory and then the application[2].

To set the setting consider the motion of a free particle in a rapidly oscillating field:

$$H(p,q) = H_0(p,q) + qF\sin\omega, t$$

where F is a constant and $H_0 = p^2/2m$ is the unperturbed Hamiltonian. It is easy to see that the equation of motion is given by

$$m\ddot{q} = -F\sin\omega t.$$

Integrating we have

$$q(t) = q_0(t) + \xi(t) = q_0(t) - \frac{F\sin\omega t}{m\omega^2}$$

and

$$p(t) = p_0(t) + \eta(t) = q_0(t) + \frac{F\cos\omega t}{\omega}$$

Thus the average motion always follows the unperturbed motion given by $q_0(t) = vt$ which is linear in time. Without further proof we may in general assume that

$$\begin{aligned} q(t) &= q_0(t) + O(1/\omega^2) \\ p(t) &= p_0(t) + O(1/\omega). \end{aligned} \tag{1}$$

in the presence of fast perturbations, that is when ω is large. Thus in the order of perturbations we assume that $\eta^2 and \xi$ are of the same order. We will use this fact presently.

Consider now a general Hamiltonian with a perturbing field that is rapidly oscillating. We want to find the Hamiltonian of mean motion. Let

$$H(p,q,t) = H_0(p,q) + V(q)\sin\omega t, \tag{2}$$

where $V(q)\sin\omega t$ is an external perturbation. In the general the soluton may be written as a combination of the smooth and oscillating part:

$$q(t) = q_0(t) + \xi(t), \quad p(t) = p_0(t) + \eta(t),$$

where we assume that the oscillating has a mean which is zero.

The Hamiltonian equations of motion are

$$\dot{q}(t) = \frac{\partial H}{\partial p} = \dot{q}_0(t) + \dot{\xi}(t) = \frac{\partial H_0}{\partial p},$$

[1]This is a statement made by Feynman in his Lectures on Physics.

[2]See Percival and Richards, Introduction to Dynamics for more details

$$\dot{p}(t) = -\frac{\partial H}{\partial p} = \dot{p}_0(t) + \dot{\eta}(t).$$

Now consider making a Taylor expansion of the full Hamiltonian given in Eq.2 in two variables around the variables of the mean motion (p_0, q_0):

$$H_0(p,q) = H_0(p_0,q_0) + \frac{\partial H}{\partial p_0}\eta + \frac{\partial^2 H_0}{\partial p_0^2}\frac{\eta^2}{2} + \frac{\partial H_0}{\partial q_0}\xi + \cdots$$

and

$$V(q) = V(q_0) + \frac{\partial V}{\partial q_0}\xi + \cdots ,$$

where we have used the fact that ξ and η^2 are of the same order.

The equations of motion to first order in ξ are given by

$$\dot{q} = \dot{q_0} + \dot{\xi} = \frac{\partial H_0}{\partial p_0} + \frac{\partial^2 H}{\partial p_0^2}\eta + \frac{\partial^3 H_0}{\partial p_0^3}\frac{\eta^2}{2} + \frac{\partial^2 H_0}{\partial q_0 \partial p_0}\xi + \cdots$$

$$\dot{p} = \dot{p_0} + \dot{\eta} = -[\frac{\partial H_0}{\partial q_0} + \frac{\partial^2 H}{\partial p_0 \partial q_0}\eta + \frac{\partial^3 H_0}{\partial p_0^2 \partial q_0}\frac{\eta^2}{2} + \frac{\partial^2 H_0}{\partial q_0^2}\xi] - [\frac{\partial V(q_0)}{\partial q_0} + \frac{\partial^2 V}{\partial q_0^2}\xi]\sin\omega t] + \cdots$$

In order to get the mean motion we average over the period of rapid oscillations with $\langle\xi\rangle = 0 = \langle\eta\rangle$. We have

$$\langle\dot{q}\rangle = \frac{\partial H_0}{\partial p_0} + \frac{\partial^3 H_0}{\partial p_0^3}\langle\frac{\eta^2}{2}\rangle + \cdots$$

$$\langle\dot{p}\rangle = -\frac{\partial H_0}{\partial q_0} - \frac{\partial^3 H_0}{\partial p_0^2 \partial q_0}\langle\frac{\eta^2}{2}\rangle - \frac{\partial^2 V}{\partial q_0^2}\langle\xi \sin\omega t\rangle + \cdots$$

The equation of motion of the oscillatory terms keeping terms up to the leading order only is given by

$$\dot{\xi} = \frac{\partial^2 H}{\partial p_0^2}\eta + \cdots$$

and

$$\dot{\eta} = -\frac{\partial V(q_0)}{\partial q_0}\sin\omega t + \cdots ,$$

Assuming q_0, p_0 vary very little over a period, the approximate solutions for the oscillating parts, $\xi(t), \eta(t)$ from the above equations are therefore given by

$$\eta(t) = \frac{\cos\omega t}{\omega}\frac{\partial V(q_0)}{\partial q_0}$$

and

$$\xi(t) = \frac{\sin\omega t}{\omega^2}\frac{\partial V(q_0)}{\partial q_0}\frac{\partial^2 H}{\partial p_0^2}\eta$$

Further we also have, for the averages,

$$\langle\xi(t)\sin\omega t\rangle = \frac{1}{2\omega^2}\frac{\partial V(q_0)}{\partial q_0}\frac{\partial^2 H}{\partial p_0^2}\eta$$

and

$$\langle \eta^2(t) \rangle = \frac{1}{2\omega^2} (\frac{\partial V(q_0)}{\partial q_0})^2$$

We now substitute these results in the equations of mean motion:

$$\langle \dot{q} \rangle = \frac{\partial H_0}{\partial p_0} + \frac{1}{4\omega^2} (\frac{\partial V(q_0)}{\partial q_0})^2 \frac{\partial^3 H_0}{\partial p_0^3}$$

and

$$\langle \dot{p} \rangle = -\frac{\partial H_0}{\partial q_0} - \frac{1}{4\omega^2} (\frac{\partial V(q_0)}{\partial q_0})^2 \frac{\partial^3 H_0}{\partial p_0^2 \partial q_0} - \frac{1}{2\omega^2} \frac{\partial V(q_0)}{\partial q_0} \frac{\partial^2 H}{\partial p_0^2} \frac{\partial^2 V}{\partial q_0^2}$$

Both these equations may be combined and written in the form of Hamiltonian equations albeit with a new Hamiltonian K,

$$\langle \dot{q} \rangle = \frac{\partial K(q_0, p_0)}{\partial p_0}$$

and

$$\langle \dot{p} \rangle = -\frac{\partial K(q_0, p_0)}{\partial q_0}$$

where

$$K(q_0, p_0) = H_0 + \frac{1}{4\omega^2} (\frac{\partial V(q_0)}{\partial q_0})^2 \frac{\partial^2 H_0}{\partial p_0^2} \tag{3}$$

which is the desired effective Hamiltonian of mean motion. Note the correction is of the order $1/\omega^2$.

2 Pendulum with a vibrating pivot

Consider a pendulum consisting of a mass m attached to a light stiff rod which is free to move vertically about the pivot. We choose $z-$ axis along the vertical and $x-$ axis tobe horizontal. If the length of the rod is l, then the coordinates of the bob are given by

$$x = l \sin \psi, \quad y = -l \cos \psi - F(t)$$

where $F(t)$ is the time dependent vertical displacement of the rod about the pivot and ψ is the generalised coordinate which is the angle between the rod and the down vertical. We will choose a specific form for this later. The potential energy is then given by

$$V(z, t) = mgz = -mg[l \cos \psi + F(t)]$$

The Lagrangian of the system is given by

$$L = \frac{1}{2} m[\dot{x}^2 + \dot{z}^2] - V(z, t).$$

Substituting for the velocities,

$$L = \frac{1}{2} m[l^2 \dot{\psi}^2 - 2l \dot{\psi} \dot{F} \sin \psi] + mgl \cos \psi + \frac{1}{2} m \dot{F}^2 + mgF$$

The last two terms are independent of ψ and its derivative and are functions of time only. They may be ignored as far as dynamics of the system is concerned.

We can further manipulate the Lagrangian by using the following identity:

$$\frac{d}{dt}[\dot{F}\cos\psi] = -\dot{F}\dot{\psi}\sin\psi + \ddot{F}\cos\psi.$$

Substituting the above in the Lagrangian above

$$L = \frac{1}{2}ml^2\dot{\psi}^2 + m(g - \ddot{F})l\cos\psi,$$

where we have ignored the total time derivative.

The conjugate momentum is given by

$$p = ml^2\dot{\psi}$$

and the Hamiltonian is

$$H = \frac{p^2}{2ml^2} - ml(g - \ddot{F})\cos\psi$$

This is indeed a nice form since the forced vertical movement can at best alter the acceleration, hence shift g.

Now consider the case when the F is given by an oscillating form:

$$F(t) = A\sin\omega t.$$

This is the same form as the rapidly oscillating perturbation that we considered in the general theory in the previous section. The Hamiltonian may be written in the form

$$H = \frac{p^2}{2ml^2} - mlg(1 + \frac{A\omega^2}{g}\sin\omega t)\cos\psi$$

solving this explicitly we may consider the Hamiltonian of the mean motion K given in Eq.3. The additional potential due to vibrations is given by

$$V(q)\sin\omega t = -ml\omega^2 A\sin\omega t.$$

Substituting this in Eq.3 we have

$$K(p_0, \psi) = \frac{p_0^2}{2m} - mgl[\cos\psi - k\sin^2\psi] = \frac{p_0^2}{2m} + V_{eff}(\psi) \tag{4}$$

where

$$k = \frac{A^2\omega^2}{4gl}.$$

The fixed points of the system are given by $p_0 = 0$ and $\psi = 0, \pi, \text{and}, \cos\psi = -1/2k$ where we have the first two are the old fixed points of the pendulum and the third one is due to the effective potential.

- At $\psi = 0$ is always a stable fixed point since

$$\frac{d^2V_{eff}}{d\psi^2} = mgl(2k+1) > 0.$$

  This is ofcourse expected.

- At $\psi = \pi$ we have

$$\frac{d^2V_{eff}}{d\psi^2} = mgl(2k-1)$$

  Thus the fixed point is always stable if $2k > 1$ or equivalently

$$A^2\omega^2\rangle 2gl,$$

  that is for fast enough oscillations the originally unstable fixed point can be made stable.

- At $\cos\psi = -1/2k$ we have

$$\frac{d^2V_{eff}}{d\psi^2} = mgl(\frac{1}{2k} - 2k)$$

  which is always unstable since for any real ψ we have $|2k| \leq 1$.

Thus indeed the unstable fixed point of the unperturbed pendulum may be stabilised provided the oscillations are fast enough.