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CLASSICAL PERTURBATION THEORY

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Books that were recommended:

- An introduction to Classical Dynamics by Ian Percival and Derek Richards
- Classical Dynamics:A Contemporary Approach by Jose and Saletan
- Other than this my proffesors notes and a walk through the papers on the proof of KAM theorem by G.Gallavotti.

As prerequisites it's better to have read the Chapter 2 of Jose and Saletan. And should know some basic differential equations and power series. Number Theory is helpful in KAM theorem.

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1. Introduction

Most of the Hamiltonians that we come accross in day-to-day life are not solvable so one of the most effective and widely used method is that we write this Hamiltonian as sum of a solvable hamiltonian and a small perturbation added to this hamiltonian(necessarily small).i.e say if our hamiltonian is H and the solvable hamiltonian as H_0 and the perturbation as ϵH_1.

$$H(q,p) = H_0(q,p) + \epsilon H_1(q,p)$$

The Central Idea of perturbation theory is to expand the solution as a power series in ϵ, similar to a Taylor expansion and approximate to the order required.

1.1. Types of Perturbation series.

(1) *Regular Perturbations*:
The number of roots or the solution for a particular equation we have does not change when ϵ is 0 .i.e say we take a simple quadratic eqn $x_2 + x - \lambda\epsilon$. This has two roots which either by expanding after you find solution or substitue x as power series givesthe same roots i.e

$$x_1 = \lambda\epsilon - \lambda_2\epsilon_2 - ....$$

$$x_2 = -1 - \lambda\epsilon + \lambda_2\epsilon_2 - ...$$

(2) *Singular Perturbations*:
Consider the follwing equation $\epsilon x_2 + x - 1$. Then if we keep ϵ as 0 we get only one root otherwise two roots. i.e for $\epsilon = 0$ and for small ϵ the limits are different. We can convert it to regular perturbation by replacing x as $x = \frac{y}{\epsilon}$.

1.2. **Regular perturbations of differential Equantions.** Lets' look into our topic through this example....

$$\frac{dx}{dt} = x + \epsilon x^2$$

where $\epsilon \ll 1$; $x(t=0) = A$ is the intitial condition. Let

$$x(t) = x_0(t) + \epsilon x_1(t) + ... = \sum_{n=0}^{\infty} \epsilon^n x_n(t)$$

$$\dot{x}(t) = \sum_{n=0}^{\infty} \epsilon^n \dot{x_n t}) = \sum_{n=0}^{\infty} \epsilon^n x_n(t) + \epsilon \sum_{n=0}^{\infty} \epsilon^n x_n(t) \sum_{n'=0}^{\infty} \epsilon^{n'} x_{n'}(t)$$

On comparing the terms we have that

$$x_0(t) = Ae_t; A = x_0(0)$$

$$x_1(t) = A^2 e^t(e^t - 1)$$

$$x_2(t) = A^3 e^t(e^t - 1)^2$$

hence our perturbative solution looks like

$$x(t) = Ae^t[1 + \epsilon A(e^t - 1) + \epsilon^2 A^2 (e^t - 1)^2$$

This is a geometric series hence turns out ot be

$$x(t) = \frac{Ae^t}{1 - \epsilon A(e^t - 1)}$$

This is the exact solution of the differential equation. *The power series expansion becomes invalid after when in the denominator $\epsilon A(e^t - 1) = 1$ i.e a singularity appears.Hence power series expansion cannot be used anymore for the time beyond this t_c. Ther reason is that the expansion does not have a small parameter anymore.* For further examples refer Percival where he discusses about a quartic oscillatorand brings out the problem with the regular perturbatrion theory and it's resolution through the conception of writing a canonical transformation from (q,p) to (I, θ). Let's directly look into the problem we face with the quartic oscillator in a simple literative way. We get our first order appx. of the solution to be

$$x(t) = Acos\omega t - \frac{\epsilon A^3}{8{\omega_0}^2}[3\omega_0 tsin\omega_0 t + \frac{(cos\omega_0 t - cos3\omega_0 t)}{4}]$$

This satisfies the intital condition but a problem is lurking here. It is that the linear dependence in the first term implies that the oscillation amplitude keeps growing without bound for any ϵ which is against the known fact that motion of a quartic oscillator is bounded for a given energy E. The reason for the failure is that the driving force on the undamped oscillator also has the same frequency ω_0. The solution breaks down if we take that $\omega_0 = \Omega$. We can get away from the difficulty by including the two expansions i.e of ω_0 also in the same parameter i.e. ϵ because the quartic term also changes ω_0. Write $\omega = \omega_0 + \epsilon\omega_1 + ...$ Even here we get a drivin term which can lead to a divergent amplitude unless we choose $\omega_1 = \frac{3A^2}{8\omega_0}$. Better and more elegant way is the *Canonical Perturbation Theory.*

2. Canonical Perturbation Theory

Let's see the first order theory here. Consider a hamiltonian

$$H(q, p) = H_0(q, p) + \epsilon H_1(q, p)$$

Where H_0 is integrable and H_1 is a perturbation. Let's look into a special example for our current study, the vertical pendulum:
$H(\psi, p) = \frac{p^2}{2} - \alpha^2 cos\psi$ where $\alpha^2 = \frac{g}{L}$. Take m=1. Here we come accross two cases because of the separatrix.

(1) Fast rotation where $\frac{p^2}{2} \gg \alpha^2$,here we can choose our $H_0$and perturbation as

$$H_0 = \frac{p^2}{2}$$
$$\epsilon H_1 = -\alpha^2 cos\psi$$

(2) Librations near the ellliptic fixed point where we can write our Hamiltonian as

$$H(\psi, p) = \frac{(p^2 + \alpha^2\psi^2)}{2} - \alpha^2 - \alpha^2(cos\psi - 1 + \frac{\psi^2}{2})$$

Here ψ is the small parameter.

Let's continue with the the general analysis of our problem.
Suppose (I, θ) are the action-angle variables of H, then $(p, q) \rightarrow (I, \theta)$ is a Canonical Transforamtion. $H(I, \theta) = K(I)$. Hence

$$\dot{I} = -\frac{\partial K}{\partial \theta} = 0$$

$$\dot{\theta} = \frac{\partial K}{\partial I} = \omega(I) \Rightarrow \theta(t) = \omega(I)t + \delta$$

Suppose J and ϕ are the action angl variables of the unperturbed Hamiltonian. This changes of variables is also a canonical transformation and $H_0(p, q) \rightarrow H_0(J)$ where

$$\dot{J} = 0; \dot{\phi} = \frac{\partial H_0}{\partial J}$$

i.e we are trying to write our hamiltonian as $H(J, \phi) = H_0(J) + \epsilon H_1(J, \phi)$, we have

$$\dot{J} = -\epsilon\frac{\partial H_1}{\partial \phi}$$

$$\dot{\phi} = \frac{H_0(J)}{\partial J} + \epsilon\frac{\partial H_1}{\partial J}$$

So we are interested in the canonical transformation of $(J, \phi) \rightarrow (I, \theta)$. We shall answer this using peerturbation theory.Let's write

$$\phi(I, \theta) = \phi_0(I, \theta) + \epsilon\phi_1(I, \theta) + ...$$
$$J(I, \theta) = J_0(I, \theta) + \epsilon J_1(I, \theta) + ...$$

All ϕ_n and J_n are independent of ϵ. Implicit assumption is that even though H_1 is a first order perturbation in ϵ, ϕ and J may involve higher order terms(comes for Singular perturbation Theory).
We can write

$$K(I) = H_0(J_0) + \epsilon H_1(J_0 + \epsilon J_1, \phi_0 + \epsilon\phi_1)$$

When $\epsilon = 0$ then $I = J_0$ and $\theta = \phi_0$,hence we get the K as

$$K(I) = H_0(I + \epsilon J_1) + \epsilon H_1(I + \epsilon J_1, \theta + \epsilon\phi_1)$$

All we need to do is try and determine J_1 and ϕ_1.We have

$$K(I) = H_0(I) + \epsilon\frac{\partial H_0}{\partial I}J_1 + \epsilon H_1(\theta, I)$$

Therefore we get $K_0(I) = H_0(I)$ and $K_1(I) = \frac{\partial H_0}{\partial I}J_1 + H_1(\theta, I)$. Here we donot know only J_1. Let's try to get it which gives us what we want. Here we need to use the area preservation property of the canonical transforamtion i.e.

$$\int d\theta K_1(t) = \int [J_1(\theta, I)\frac{\partial H_0}{\partial I} + H_1(\theta, I)]d\theta$$

on using the property that $(I, \theta) \rightarrow (J, \phi)$ is a Canonical Transforamtion and the area preserving property we get

$$K_1(I) = \frac{1}{2\pi}\oint H_1(\theta, I)d\theta = \frac{\oint H_1(\theta, I)d\theta}{\oint d\theta} \equiv \langle H_1(\theta, I)\rangle$$

Therefore first order correction is just the mean of H_1 taken over the unperturbed motion.Thus

$$K(I) = H_0(I) + \epsilon\langle H_1(I)\rangle$$

From this we can get our J_1which turns out to be

$$J_1 = \frac{\langle H_1(I)\rangle - H_1(I, \theta)}{\omega_0(I)}$$

For perturbation to be valid the frequency $\omega_0(I)$ should *not* be small as it happens near the separatrix. to find ϕ we can use the fact that the Jacobian $\frac{\partial(\phi, J)}{\partial(\theta, I)} = 1$ where we get as

$$J = 1 + \epsilon\left\{\frac{\partial\psi_1}{\partial\theta} + \frac{\partial J_1}{\partial I}\right\} \Rightarrow \frac{\partial\phi_1}{\partial\theta} = -\frac{\partial J_1}{\partial I}$$

2.1. **The problem with the Canonical Perturbation theory.** Consider $\vec{m}.\vec{\omega_0} = \sum_k m_k\omega_{0k}$ where m_k are set of integers. O when $\omega_k' s$ are commensurate i.e. resonant or degenerate, then there always exists a set of m's such that $\vec{m}.\vec{\omega_0} = 0$. Canonical perturbation theory *fails* because the sum diverges even in first order. In fact even when ω_k's are not commensurate, the sum may become arbitrarily small (this is bound to happen). This infact is the content of the "small divisors" problem that plagued the adance of Mechanics for almost a century. We will be looking into this at a later stage.
Thus perturbation may work only in a restricted section and for extremely small ϵ.When it works the system are completely integrable and the invariant tori of the unperturbed system are slightly distorted. When it does not tori of the original state are totally destroyed.

The main problem with CPT is that it assumes the perturbed system to be completely integrable. We use this fact in averaging to get K. This forces that integrability be valid at every order of perturbation theory – kind of a self fulfilling strategy.

Refer Physical Review Vol.188 page 416 for further reading on Destruction of Tori.

3. Adiabatic and Rapidly Oscillating Conditions

When the Hamiltonian depends explicitly on time the cases are not so simple as we have found till now; because a general analysis of the motion does not exist. However the motion of related conservative systems can be used to get apprximate solutions. Two situations arise where these approximations are very helpful.

- If a motion is approximately periodic and the Hamiltonian changes very little during a period.The slow changes are called **adiabatic**.
- If the system is acted upon by external periodic force of small period when compared to the time in which the unperturbed motion changes significantly. This is the case of **rapid oscillations**

Let's first look into the Adiabatic Conditions.

3.1. Adiabatic Conditions.

3.1.1. *An examplary introduction.* Consider an *elastic* ball moving between two moving planes. We know that a particle in a box at each collsions the momentum or velocity changes discontinously i.e. $p \to -p$. Here the Hamiltonian is $H = \frac{p^2}{2m}; p = \pm\sqrt{2mE}$. Hence the action is $I = \frac{mvx\sqrt{E}}{\pi}$ i.e for $m = 1 and E = 1$ we get action to be $I = \frac{vx}{\pi}$
Now let's come to the case of moving planes, say they move at a velcity V relative to each other such that $0 \leq |V| \ll |v|$.For simplicity we can consider one of the palte to be fixed and other to be oscillating with a velocity V. Our convention will be positive for moving towards the plate and negative otherwise.
Since $|V| \ll v$, the palnes move very little between succesiv collisons of th ball hence,

$$\Delta t \approx \frac{2x}{v}$$

if v is velocity before collsion and v' is the velocity after collision then $v' = v + 2V$, because the relative velocity $v' - V = v + V$(relative to centre of mass).

$$\Rightarrow \frac{dv}{dt} = \frac{v' - v}{\Delta t} = \frac{V.v}{x}$$

Therefore the change in action is

$$\pi \frac{dI}{dt} = x\frac{dv}{dt} + v\frac{dx}{dt} = V.v - v.V = 0$$

Thus the action is approximately constant because as v increases x decreases and vice-versa.
This can be seen precisely: Say v_{n+1} is velocity after nth collision, then $v_{n+1} = v_n + 2V \Rightarrow v_{n+1} = v_0 + 2nV$.Say x_n is the separation at the instant of collision. Then $x_{n+1} - x_n = -V\Delta t_n \Rightarrow x_{n+1} = \left\{\frac{1+z_n}{1+3z_n}\right\} x_n$ where $z_n = \frac{V}{v_0+2nV}$. Hence we get our action to be $\pi I_{n+1} = v_{n+1}x_{n+1} \Rightarrow I_{n+1} = \left\{1 + \frac{2{z_n}^2}{1+3z_n}\right\} I_n$. Let's take some initial conditions such that Adiabatic condition is satisfied: $v_0 = 1; I_0 = \frac{1}{\pi}; V = 0.01$ As $x_n \to 0$as $n \to \infty$ and $E_n \to \infty$ as $n \to \infty$, I_n is close to $\frac{1}{\pi} = I_0$. If the planes are relatively moving apart then amplitude x increase but the energy decreases such

that action is almost constant. One of the application is tp a gas in a box of moving walls.

3.1.2. *General Adiabatic Theory.* Consider a Hamiltonian $H(q,p,\lambda)$ where λ is the parameter. If λ is constant each value os it defines a unique Hamiltonian for which we may assume the motion to be bounded and there exist action-angle variable (I,θ). If λ is slowly varying in t we can set it as $\lambda = \epsilon t$ where $0 < \epsilon < 1$, where ϵ is fixed such that the Hamiltonian changes significantly only over the variations of order $\lambda \sim 1$. Then $\frac{\partial H}{\partial t} = \epsilon \frac{\partial H}{\partial \lambda}$. Consider a function $F(t) = F(q(t), p(t), \lambda(t))$. F is an "adiabatic invariant" if for any ϵ, F varies little during the interval $0 < t < \frac{1}{\epsilon}$. Precisely, for any $\eta > 0$, there exists ϵ_0 such that $0 < \epsilon < \epsilon_0, \lambda = \epsilon t$ and $|F(t) - F(0)| < \eta$ then F is a adiabatic invariant.Now let's try to apply to variation of I:
We know that $(q,p) \to (I,\theta)$ is a canonical transforamtion where θ is bounded and I and θ are genuinely action and angle variables. The genetrating function $F_1(\theta, q, \lambda)$, where λ is a parameter. If λ varies it is still a Canonical Transforamtion but I and θ are not action and angle variables (This section can be solved using Lie Perturbtion Theory).

$$K(I,\theta,\lambda) = H(q,p,\lambda) + \epsilon \frac{\partial F_1}{\partial \lambda}$$

$$\dot{\theta} = \omega(I,\lambda) + \epsilon R_1(\theta, I, \lambda); \dot{I} = -\epsilon R_2(\theta, I, \lambda)$$

where

$$R_1 = \frac{\partial^2 F_1}{\partial I \partial \lambda}; R_2 = \frac{\partial^2 F_1}{\partial \theta \partial \lambda}$$

If we take F_1 to be periodic θ (bounded motional then the mean of R_2 is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 F_1}{\partial \lambda \partial \theta} d\theta = 0$$

As F_1 is peiodic in θ, R_1 is also necessarily periodic. We now want to prove, that for soem $\eta < 0$,

$$|I(t) - I(0)| < \eta; 0 < t < \frac{1}{\epsilon}$$

such that $I(t) \approx I(0)$. Now consider an auxilliary variable R_3 as

$$R_3(\theta, I\lambda) = \int_0^{\theta} R_2(\theta', I, \lambda) d\theta'$$

Here R_3 is also periodic. Therefore we get that

$$\frac{d}{dt}\left\{\frac{R_3}{\omega}\right\} = \frac{1}{\omega}\frac{\partial R_3}{\partial \theta}\dot{\theta} + O(\epsilon) terms$$

$$R_2 = \frac{d}{dt}\left\{\frac{R_3}{\omega}\right\} - \epsilon M(t)$$

which gives

$$\dot{I} = -\epsilon \frac{d}{dt}\left\{\frac{R_3}{\omega}\right\} + \epsilon^2 M(t)$$

As R_1, R_2, R_3 are periodic and $\omega \neq 0$, $M(t)$ is bounded. Hence we have th t

$$|I(t) - I(0) \leq 2\epsilon M_2 + \epsilon^2 t M_1; |M(t)| \leq M_1$$

$$\Rightarrow |I(t) - I(0) \leq K\epsilon$$

i.e. $I(t)$ remains close to $I(0)$ in the interval $0 < t < \frac{1}{\epsilon}$

3.2. **Motion in a rapidly oscillating field : Fast perturbations.** Consider for example the motion of a free particle in a rapidly oscillating field:

$$H(p,q) = H_0(p,q) + qF\sin\omega t; Fisaconstant$$

$$\Rightarrow \dot{q} = \frac{p}{m}$$

$$\dot{p} = -F\sin\omega t \Rightarrow m\ddot{q} = -F\sin\omega t$$

On inegrating this we get

$$q(t) = q_0(t) + \frac{F\sin\omega t}{m\omega^2}; p(t) = p_0(t) - \frac{F\cos\omega t}{\omega}$$

where $q_0(t) = vt$ is linear in time. Therefore, the average motion follows the unperturbed motion given by $q_0(t)$. Here we have to note that $q(t) = q_0)t) + O(\frac{1}{\omega^2}); p(t) = p_0(t) + O(\frac{1}{\omega^2})$ (these are true in general). We shall use this form in general for any Hamiltonian perturbed by a rapid oscillation.
Now, consider a general Hamiltonian(we want to find a Hamiltonian of mean motion).

$$H(q,p) = H_0(q,p) + V(q)\sin\omega t$$

where $V(q)\sin\omega t$ is an external perturabtion. In general the perturbation may be written as,

$$q(t) = q_0(t) + \xi(t); p(t) = p_0(t) + \eta(t)$$

hence we can get our $\dot{q}$ and $\dot{p}$ and we get our Hamiltonian to be

$$\begin{aligned} H_0(q,p) &= H_0(q_0 + \xi, p_0 + \eta) \\ &= H_0(q_0,p_0) + \frac{\partial H_0}{\partial p_0}\eta + \frac{\partial^2 H_0}{\partial {p_0}^2}\frac{\eta^2}{2} + \xi\frac{\partial H_0}{\partial q_0} + .. \end{aligned}$$

here we used the fact that ξ and η^2 are of the same order. Therefore we have,

$$V(q) = V(q_0) + \xi\frac{\partial V}{\partial q_0} + ...$$

Hence we have

$$\dot{q} = \dot{q_0} + \dot{\xi} = \frac{\partial H_0}{\partial p_0}\left\{\eta\frac{\partial^2 H_0}{\partial {p_0}^2}\right\}^{\$} + \frac{\eta^2}{2}\frac{\partial^3 H_0}{\partial {p_0}^3} + \xi\frac{\partial^2 H_0}{\partial p_0 \partial q_0}$$

$$\dot{p} = \dot{p_0} + \dot{\eta} = -\left\{\frac{\partial H_0}{\partial q_0}\eta\frac{\partial^2 H_0}{\partial p_0 \partial q_0} + \frac{\eta^2}{2}\frac{\partial^3 H_0}{\partial {p_0}^3 \partial q_0} + \xi\frac{\partial^2 H_0}{\partial {q_0}^2}\right\} - \left\{\left\{\frac{\partial V}{\partial q_0} + \xi\frac{\partial^2 V}{\partial {q_0}^2}\right\}\sin\omega t\right\}^{\$}$$

Consider the mean motion averaged over the period of the rapid oscillations (shown by \$). i.e us $\langle\xi\rangle = \langle\eta\rangle = 0$

$$\langle\dot{q}\rangle = \frac{\partial H_0}{\partial p_0} + \frac{1}{2}\langle\eta^2\rangle\frac{\partial^3 H_0}{\partial {p_0}^3}$$

$$\langle\dot{p}\rangle = -\frac{\partial H_0}{\partial q_0} - \langle\xi\sin\omega t\rangle\frac{\partial^2 V}{\partial {q_0}^2} - \frac{1}{2}\langle\eta^2\rangle\frac{\partial^3 H_0}{\partial q_0 \partial {p_0}^2}$$

keeping only the leading oscillatory terms, gives us

$$\dot{\xi} = \eta\frac{\partial^2 H_0}{\partial {p_0}^2} \Rightarrow \xi(t) = \frac{\sin\omega t}{\omega^2}\frac{\partial V}{\partial q_0}\frac{\partial^2 H_0}{\partial {p_0}^2}$$

$$\dot{\eta} = -\frac{\partial V}{\partial q_0}\sin\omega t \Rightarrow \eta(t) = \frac{\cos\omega t}{\omega}\frac{\partial V}{\partial q_0}$$

Assuming (q_0, p_0) to vary very slowly over one period

$$\langle\xi\sin\omega t\rangle = \frac{1}{2\omega^2}\frac{\partial V}{\partial q_0}\frac{\partial^2 H_0}{\partial {p_0}^2}; \langle\eta^2\rangle = \frac{1}{2\omega^2}\left\{\frac{\partial V}{\partial q_0}\right\}^2$$

Using this we obtain the Hamiltonian of mean motion as $\langle\dot{q}\rangle$ and $\langle\dot{p}\rangle$,

$$\langle\dot{q}\rangle = \frac{\partial}{\partial p_0}\left\{H_0 + \frac{1}{4\omega^2}\left\{\frac{\partial V}{\partial q_0}\right\}^2 \frac{\partial^2 H_0}{\partial {p_0}^2}\right\}$$

$$\langle\dot{p}\rangle = -\frac{\partial}{\partial q_0}\left\{H_0 + \frac{1}{4\omega^2}\left\{\frac{\partial V}{\partial q_0}\right\}^2 \frac{\partial^2 H_0}{\partial {p_0}^2}\right\}$$

Thus the Hamiltonian of the mean motion is given by

$$K(q_0, p_0) = H_0 + \frac{1}{4\omega^2}\left\{\frac{\partial V}{\partial q_0}\right\}^2 \frac{\partial^2 H_0}{\partial {p_0}^2}$$

We have to note here that the correction is of the order $\frac{1}{\omega^2}$.