VERTICALLY AND HORIZONTALLY DRIVEN PENDULUMS

RAVITEJ UPPU

ABSTRACT. In this I will be doing two cases majorly where the pendulum's support moves horizontally and vertically with a frequency ν.

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Chennai Mathematical Institute.

1. Pendulum with a vibtrating support

Definition Of A Fixed Point : Here we see that the systems are also dependent on time. Thus our fixed ppoints here are those whose ψ and $\dot{\psi}$ are null at a time t when the particle reaches that point in phase space given a set of initial conditions.

1.1. Vertically Driven Pendulum.

In all the cases our pendulum has a light stiff rod to which a mass m is connected. Let the distance of support from origin O be given by the function $\mathbf{D(t)}$. Then our x and y will be the distance functions of the pendulum

$$x = lsin\psi \Rightarrow \dot{x} = (l\dot{\psi}cos\psi)$$
$$y = -lcos\psi - D(t) \Rightarrow \dot{y} = (l\dot{\psi}sin\psi - \dot{D}(t))$$

Hence

$$v^2 = (\dot{x}^2 + \dot{y}^2)$$
$$\Rightarrow v^2 = (l^2\dot{\psi}^2\cos^2\psi + (l\dot{\psi}sin\psi - \dot{D}(t))^2)$$

We have
Potential Energy $\mathbf{V} = -mg(lcos\psi + D(t))$
Kinetic Energy $\mathbf{T} = \frac{1}{2}m(l^2\dot{\psi}^2\cos^2\psi + (l\dot{\psi}sin\psi - \dot{D}(t))^2)$

$$\Rightarrow T = \frac{1}{2}m(l^2\dot{\psi}^2) + \dot{D}^2 - 2l\dot{\psi}sin\psi\dot{D})$$

Let's write the Lagrangian

$$\mathbf{L} = \mathbf{T} - \mathbf{V}$$
$$\Rightarrow L = \frac{1}{2}m((l^2\dot{\psi}^2) + \dot{D}^2 - 2l\dot{\psi}sin\psi\dot{D}) + mg(lcos\psi + D(t))$$

But we know that **any two Lagrangians which differ by a total differential of time give same equations of motion**. Hence we can remove the terms $\dot{D}^2$ and $mgD(t)$. Hence my Lagrangian becomes

$$L = \frac{1}{2}m(l^2\dot{\psi}^2 - 2l\dot{\psi}sin\psi\dot{D}) + mglcos\psi$$

Let's do a small trick here. Let's replace $\dot{\psi}\dot{D}sin\psi$ by $(\ddot{D}cos\psi - \frac{d}{dt}(\dot{D}cos\psi))$.
We get

$$L = \frac{1}{2}m(l^2\dot{\psi}^2 - 2l(+\ddot{D}cos\psi - \frac{d}{dt}(\dot{D}cos\psi)) + mglcos\psi)$$

Therefore

$$\mathbf{L} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{ml^2}\dot{\psi}^{\mathbf{2}} + \mathbf{ml(g - \ddot{D})cos}\psi$$

Behold this is like a *time-variant gravitational field*. And we already know the first part of the equation which is *simple pendulum (i.e when* $\ddot{D} = 0$).

Now let's try to get the equation of motion from

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\psi}}) = \frac{\partial L}{\partial \psi}$$

Hence on calculating we have

$$\ddot{\psi} = \frac{\dot{\psi}}{\mathbf{l}}\mathbf{sin}\psi(\mathbf{\ddot{D}} - \mathbf{g})$$

Here we have that ψ and $\dot{\psi}$ are the variables. Let's try to analyze this motion putting $\psi = R$ and $\dot{\psi} = S$. From this we have that

$$\dot{R} = S$$
$$\dot{S} = \frac{1}{l}(\ddot{D} - g)sinR$$

At the fixed points $\dot{R} = \dot{S} = 0$ from this we get that $S = 0$ and $R = n\pi$

Therefore we can compute the A-matrix to be

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{l}(\ddot{D} - g)cosR & 0 \end{pmatrix}$$

This gives us

$$\lambda^2 = -\frac{1}{l}(\ddot{D} - g)cosR$$

At the point $(\pi, 0)$

$$\lambda^2 = -\frac{1}{l}(g - \ddot{D})$$

So unless $g > \ddot{D}$ this not negative hence we get that at the point $(\pi, 0)$ if the particle reaches at a time t_π we get the point to be an elliptic point and hence STABLE

REFERENCE: *This can be found as an introduction of the paper written by M.V.Bartuccelli,G.Gentile, K.V.Georgiou with topic "On the dynamics of Vertically Driven Damped Planar Pendulum" submitted in August 2000.A reference can be found at "www.maia.ub.es/cgi-bin/mps?key=99-235".*

1.2. Horizontally Driven Pendulum. Let's look at what happens when we have a horizontal driving force now.Now our x and y will be

$$x = lsin\psi - D(t) \Rightarrow \dot{x} = l\dot{\psi}cos\psi - \dot{D}$$

$$y = lcos\psi - D(t) \Rightarrow \dot{y} = -l\dot{\psi}sin\psi$$

from tjis we get T to be

$$T = \frac{m}{2}(l^2 + \dot{D}^2 - 2l\dot{\psi}\dot{D}cos\psi)$$

And we have **V**as

$$V = -mglcos\psi$$

This gives us the Lagrangian as

$$\mathbf{L} = \frac{\mathbf{ml^2}\psi^\mathbf{2}}{\mathbf{2}} + \mathbf{ml}(\ddot{\mathbf{D}}\mathbf{sin}\psi + \mathbf{gcos}\psi)$$

Hence from this by using the Euler-Lagrangian equation of motion we have that

$$\ddot{\psi} = \frac{\dot{\psi}}{\mathbf{l}}(\ddot{\mathbf{D}}\mathbf{cos}\psi - \mathbf{gsin}\psi)$$

which is just a term afar from the simple pendulum case. Now let's try to do the same analysis as above.For fixed points $\dot{R} = \dot{S} = 0$ hence we have $S = 0$ and $tanR = \frac{\ddot{D}}{g}$ which says our $R = tan^{-1}\frac{\ddot{D}}{g}$. If we compare with simple pendulum, i.e it has an unstable point at $(\pi, 0)$which says that $R = \pi$ hence we get that $\ddot{D}(t_\pi) = 0$. Now this is a fixed point of my system. Now if we try to write the A-matrix at time t_π and at point $(\pi, 0)$ we have it as

$$A_(\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{l}(\ddot{D}(t_\pi)sin\pi + gcos\pi) & 0 \end{pmatrix}$$

Therefore we get that $\lambda^2 = -\frac{g}{l}$. This is always negative HENCE MY POINT HAS BECOME STABLE UNDER THE TAKEN CONDITIONS.

2. The support moving on a circle in x-y plane

Let't take the distance function on x and y to be $D_x(t)$ and $D_y(t)$ where for a circle $D_x(t) = Asin\omega t$ and $D_y(t) = Acos\omega t$ where ω is the angular velocity. Now our x and y are

$$x = lsin\psi - D_x \Rightarrow \dot{x} = l\dot{\psi}cos\psi - \dot{D}_x$$
$$y = -lcos\psi - D_y \Rightarrow \dot{y} = l\dot{\psi}sin\psi - \dot{D}_y$$

Our Kinetic Energy **(T)** is

$$T = \frac{m}{2}(l^2\dot{\psi}^2 + \dot{D_x}^2 + \dot{D_y}^2 - 2l\dot{\psi}(\dot{D}_x cos\psi + \dot{D}_y sin\psi)$$

and Potential Energy **(V)** is

$$V = -mg(lcos\psi - D)$$

Now we write the Lagrangian and use a similar trick whick we used in 1.1 and we get **L**

$$\mathbf{L} = \frac{\mathbf{m}}{\mathbf{2}}\mathbf{l^2}\psi^{\mathbf{2}} + \mathbf{ml}(\ddot{\mathbf{D}}_{\mathbf{x}}\mathbf{sin}\psi + (\mathbf{g} - \ddot{\mathbf{D}}_{\mathbf{y}})\mathbf{cos}\psi)$$

Hence we get the Euler-Lagrangian Equation of motion as

$$\ddot{\psi} = \frac{\mathbf{1}}{\mathbf{l}}(\ddot{\mathbf{D}}_{\mathbf{x}}\mathbf{cos}\psi + (\ddot{\mathbf{D}}_{\mathbf{y}} - \mathbf{g})\mathbf{sin}\psi)$$

Here the variables are ψ and $\dot{\psi}$. Let's call them as R and S respectively. Then we get the equations as

$$\dot{R} = S$$

$$\dot{S} = \frac{1}{l}(\ddot{D}_x cosR + (\ddot{D}_y - g)sinR)$$

hence the fixed points are $S = 0$ and $tanR = \frac{\ddot{D}_x}{(g-\ddot{D}_y)}$. Here if we want $(\pi, 0)$ to be a fixed point then $\ddot{D}_x(t_\pi) = 0$ at time t_π. Hence we get the A-matrix to be

$$A_\pi = \begin{pmatrix} 0 & 1 \\ -\frac{1}{l}(\ddot{D}_x(t_\pi)sin\pi + (g - \ddot{D}_y(t_\pi)cos\pi & 0 \end{pmatrix}$$

Hence we get

$$\lambda^2 = -\frac{1}{l}(g - \ddot{D}_y(t_\pi))$$

which says that we have this point to be an elliptic fixed point when we reach this point at time t_π and only if $g > \ddot{D}_y(t_\pi)$. Hence under these conditions our point is

a stable fixed point.
Now as the motion of the support we have considered to be on the circle hence at time t_π we have $-A\omega^2 sin\omega t_\pi = 0 \Rightarrow t_\pi = \frac{2\pi}{\omega}$ and $A\omega^2 < g$ These are the conditions for our point to be a STABLE FIXED POINT.