Introductory Classical Dynamics

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1 Newtonian Formulation

This formulation is based on 3 principles :

1. Existence of special observers in which a particle not acted upon by any force moves in a *rectilinear* fashion. These are the *inertial observers*. Here, **time** can be defined such that motion is uniform. Other observers that move in rectilinear fashion w.r.t these special frames are also inertial.
2. Once time has been chosen, the motion in these special frames can be given by $m\vec{a} = \vec{F}(\vec{r}, t, \frac{d\vec{r}}{dt})$.
3. Total momentum and Angular momentum of an *isolated* system of particles are constant.

Remark: A confusion that usually occurs is the interpretation of the First principle as a special case of the second principle when the $\vec{F} = 0$. This is *not* correct. The only conclusion that can be drawn from them is that inertial frames can only be defined dynamically.

2 Space and Time in Classical Mechanics

The Newton's formulation can be ascribed to be based on three concepts which it assumes as *God-given (canonical) and absolute*. These are:

1. Notion of a *straight line*.
2. *Ability to state* that no forces are acting on a body.
3. Notion of an *absolute time* as *flowing uniformly*.

From the way the Classical Mechanics was formulated by Newton, we see that the definition of *a free body is something that is far away from any other body in the universe.* This shows the underlying roots of the Galilean definition of inertial frame, which defined an inertial observer to be a frame that is fixed w.r.t. the fixed stars. Newton's idea of abstractly generalizing this was a manifestation of his impressions on the existence of various frames through coordinate transformations.

2.1 Newton's dynamics as Comparision dynamics

The notion of a straight line needs a structure of a Vector Space on the space in which the particle's motion is being seen. The same space can be given different structures, which denote different observers. Hence, straight line is observer-dependant.

Similarly, the rate of flow of time can be recognized to be uniform only when measured againt some otehr rate of flow that is taken as a standard and hence the claim of *Comparision Dynamics.*

Following these arguments, Time Congruence can be defined through the existence of an isolated physical system that reverts to a previous state at some instant, which implies, a sequential flow of the same states over and over again repeated in time and hence, each interval is equally long which can be considered as the standard frame.

2.2 Einstein's Relativity

The Galilean frame's prescence in the Special Theory of Relativity can be felt from the situation of a light ray passing through an inertial frame. Here, the velocity of the light ray is same regardless of the motion of the source or the frame and regardless of the direction of the ray. This definition is obviously superior to the previous definition as this does not involve damping forces which come up in the situation of moving objects.

Relevance of this definition lies in the consequences that follow from the the attempts to correlate space and time between two inertial frames that are in relative motion. Here, the universality of the space and time congruence does not exist anymore. The congruence can now be only defined universally i.e. independant of the observer in the new four-dimensional space-time.

Now, for a brief look into the principle of General Theory of relativity as mentioned in the talk. This principle suggests that, for any space-time point, a locally inertial reference frame can be chosen such that in a small neighbourhood around that point, the motion of a free falling particle can be described by

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0$$

where, ξ's are the coordinates in the locally inertial frame and τ is a parametrization of the curve.

Considering, the transformation from the lab coordinates x^α to these new coordinates, we can arrive at

$$\frac{d^2x^\lambda}{d\tau^2} + \sum_{\mu,\nu=0}^{3} \left\{ \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \right\} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

This represents the motion of a particle in a gravitational field. The term in the braces are called *affine connection coefficients* and represent the gravitational force in the lab frame.

Summary

- For a particle that is moving on a space-time described by the quadruplet (x^0, x^1, x^2, x^3) (lab frame), the evolution of states is given by $\frac{d^2x^i}{d\tau^2} = F^i(x^j, \frac{dx^j}{d\tau}$.
- Motion is said to be *free* if $\exists$ a systems $(\xi^0, \xi^1, \xi^2, \xi^3)$ such that $\frac{d^2\xi^i}{d\tau^2} = 0$.
- Solution to the last equation gives the structure of the space. Thus, inertial frames are dynamically defined relatively to some chosen *comparision system*
- The system is defined to be physical iff $\frac{d\xi^0}{d\tau} \neq 0$.
- The General Theory of Relativity is not a theory of invariance like SR which tells us how to choose a reference frame or rather what to choose as a reference frame. It actually tells us how to represents forces as gravitational forces and helps us to live with them, because, the affine connection coefficients which represent the gravitational forces are not absent in the equation of motion.

3 Lagrangian and Hamiltonian Formulation

I will not write the equations of motion given through these formulations as these can be found in any book. I only add a few remarks or setup couple of definitions which I might use later.

The transition from Newtonian to these formulations is not just a mathematical manipulation to simplify things. There is a certain fact that is being presumed on entering these formulations, which is the existence of the inner product (scalar product) which plays a crucial role in the indirect measurement of the Generalized forces that appear in the Lagrangian formulation which requires a metric structure on the space after calculating the work which is the physical quantity. The reason for the necessity of the scalar product comes from the fact that work is a scalar quantity, whereas, force is a vectorial field and hence the interconnection requires the presence of a scalar product.

3.1 Concerning Constraints and Degrees of Freedom

- Aptitude of a system of particles is defined as the set of all the positions and velocities of the particles in the system.
- Contraints can be *inner* (due to the intrinsic property of body, like rigidity, etc.) or *outer* (due to the presence of obstacles).
- Contraints are *Holonomic*, when they are restrictions placed only on the positions of the system of particles and *Heteronomic*, when the restrictions are on the aptitude of the system of particles.

Consider a particle moving on a curve γ which is parametrized in a Cartesian frame by λ as $x = \phi(\lambda), z = \psi(\lambda), z = \chi(\lambda)$. γ pertains to a physical system when ϕ, ψ and χ and their derivatives are continuous, $\sqrt{\dot{\phi}^2 + \dot{\psi}^2 + \dot{\chi}^2} > 0$ and for $\lambda_1 \neq \lambda_2$, all ϕ, ψ and χ cannot be equal simultaneously. These conditions physically mean that the positions and velocities are continuous, the speed is positive and the curve can not cross itself at the same instant of time, which puts an uncertainity in the evolution of particle (not supported by Classical mechanics). A curve that satisfies all these properties can be called in a single term mathematically as being *regular*

3.2 Equivalent Lagrangians

From the action integral, we see that two Lagrangians that differ only by a term which is a total derivative in time have the same variation. Such Lagrangians are said to be *Equivalent Lagrangians*

3.3 Dynamical Similitude

We can see from the Lagrangian equations of motion that two Lagrangains that differ by a constant factor result in the same equations. This helps in inferring certain properties without resorting to the integration of equations of motion.

Let's consider a system with $T = \frac{m}{2}\sum_k \dot{q_k}^2$ and $U(\lambda q) = \lambda^f U(q)$, i.e. a homogeneous function of coordinates. By performing the transfomations,

$$q_k \rightarrow \lambda q_k \& t \rightarrow \mu t$$

we see that,

$$\begin{aligned} T &\rightarrow \left\{\frac{\lambda}{\mu}\right\}^2 T \\ U &\rightarrow \lambda^f U \\ \Rightarrow L &\rightarrow \left\{\frac{\lambda}{\mu}\right\}^2 T - \lambda^f U \end{aligned}$$

And hence, to have the final Lagrangian as a multiple of inital one, the condition is that $\mu = \lambda^{1-f/2}$.

Few examples that ca be quoted here are the independance of the time period of small oscillations on length, where $f = 2$ and Kepler's Third Law where $f = -1$ and hence gives the relation between orbital period and semi-major axis of orbit.

The implication of changing the coordinates by a factor is to pass from some trajectory to another which is *geometrically similar* to the first one, only differing in the linear dimensions.